ECO-generation for compositions and their restrictions

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A composition c of n can be written as $c = (c_1, c_2, \ldots, c_k)$ with $c_1 + c_2 + \ldots + c_k = n$ and $c_i \geq 1$, $\forall i \leq k$.

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$\mathcal{C}(n)$	is the set of compositions of an integer n
$\mathcal{C}_{\leq p}(n)$	is the set of compositions of n with all parts of sizes $\leq p$
$\mathcal{C}_{1,p}(n)$	is the set of $(1,p)$-compositions of n
$\mathcal{C}_{\hat{p}}(n)$	is the set of compositions of n without parts of size p
$\mathcal{C}_{\#p}(n)$	is the set of compositions of n with at most p parts
$\mathcal{C}_{*}(n,p,r)$	is the set of compositions of n with the last part of size $= r$ mod p
$\mathcal{C}(n,p,r)$	is the set of compositions of n with all parts of size $= r$ mod p

Some bibliographies

- Alladi and Hoggatt, *Compositions with ones and twos*, 1975.
- Carlitz, *Restricted compositions*, 1976.
- Chinn and Heubach, $(1, k)$-*Compositions*, 2003.
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- Klingsberg, *A Gray code for compositions*, 1982.
- Walsh, *Loop-free sequencing of bounded integer compositions*, 2000.
- Vajnovszki, *A loopless generation of bitstrings without* p *consecutive ones*, 2001.
- Baril and Moreira, *More restrictive Gray code for (1,p)-compositions and relatives*, 2008.

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- Barcucci et al., *ECO : a methodology for the Enumeration of Combinatorial Objects*, 1999.

Recalls

- ECO method - Generating tree
- Pattern avoiding permutations
- Active sites - Right justified sites
- Regular class - c-Regular class
- Succession functions - General generating algorithm

- The ECO method is used for the enumeration and the recursive construction of combinatorial object classes.
- This is a recursive description of a combinatorial object class which explains how an object of size n can be reached from one and only one object of inferior size.

- The ECO method is used for the enumeration and the recursive construction of combinatorial object classes.
- This is a recursive description of a combinatorial object class which explains how an object of size n can be reached from one and only one object of inferior size.
- It consists to give a system of *succession rules* for a combinatorial object class which induces a *generating tree* such that each node is labeled : the set of successions rules describes for each node the label of its successors.

Pattern

- $\mathfrak{S}_n$ - the set of permutations on $[n] = \{1, 2, \ldots, n\}$.
- Let $a = a_1 \ldots a_k$. The pattern of a is the permutation $\tau \in \mathfrak{S}_k$ obtained from a by substituting the minimum element by 1, the second minimum element by 2, . . ., and the maximum element by k.

Example

The pattern of $a = 914$ is $\tau = 312$.

Pattern Avoiding Permutation

For a $\tau \in \mathfrak{S}_k$ and a $\pi \in \mathfrak{S}_n$, π is τ-avoiding iff there is no subsequence $\pi(i_1)\pi(i_1)\ldots\pi(i_k)(i_1 < i_2 < \ldots < i_k)$ whose pattern is τ. We write $\mathfrak{S}_n(\tau)$ for the set of τ-avoiding permutations of $[n]$.

Example

- $\pi = 512634$ avoids 321-pattern.
  But π contains 3412-pattern ($\underline{5}12\underline{634}$).

A barred permutation pattern is a permutation pattern in which overbars are used to indicate that barred values cannot occur at the barred positions.

Example

$\pi = \underline{5}7\underline{1}6\underline{3}4\underline{2}$ fails to be 4132-avoiding but is $4\bar{1}32$-avoiding.

Active Sites

- The *sites* of $\pi \in \mathfrak{S}_n(T)$ are the positions between two consecutive elements, before the first and after the last element.
- The sites are numbered, from right to left, from 1 to $n+1$.
- i is an *active site* of $\pi \in \mathfrak{S}_n(T)$ if the permutation obtained from π by inserting $n+1$ into its ith site is a permutation in $\mathfrak{S}_{n+1}(T)$.

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- $\chi_T(i,\pi)$ - the number of active sites of the permutation obtained from π by inserting $n+1$ into its ith active site.

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- $\chi_T(i, \pi)$ - the number of active sites of the permutation obtained from π by inserting $n+1$ into its ith active site.
- The active sites of a permutation $\pi \in \mathfrak{S}_n(T)$ are *right justified* if the sites to the right of any active site are also active.

Example

$13452 \in \mathfrak{S}_5(312)$ has 3 first active sites right justified following $134\_5\_2\_$.

Regular pattern

A set of patterns T is called *regular* if

- $1 \in \mathfrak{S}_1(T)$ has two sons,
- all active sites are right justified,
- for any $n \geq 1$ and $\pi \in \mathfrak{S}_n(T)$, $\chi_T(i,\pi)$ does not depend on π but solely on i and on the number k of active sites of π. In this case we denote $\chi_T(i,\pi)$ by $\chi_T(i,k)$ and we call it *succession function* [Do, Vajnovszki 2007].

$$(k) \rightsquigarrow (\chi_T(1,k))(\chi_T(2,k))\ldots(\chi_T(k,k))$$

or $(k) \rightsquigarrow \cup_{i=1}^{k}(\chi_T(i,k))$, for $k \geq 1$,

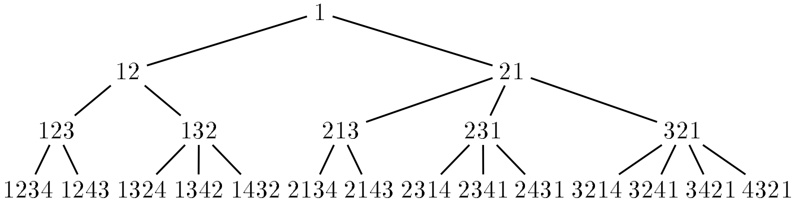
is the succession rule corresponding to the set of patterns T.

- succession function $\longrightarrow$ succession rule
- succession rule $\not\rightarrow$ succession function

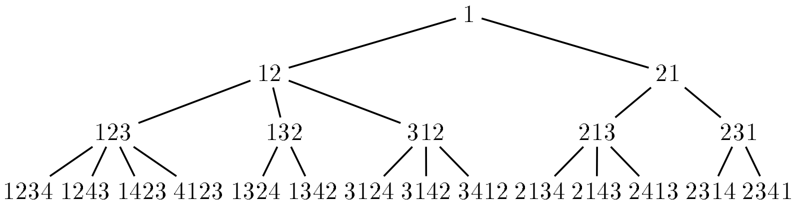
Example

The Catalan sets of permutations avoiding pattern $T = \{312\}$ and $T = \{321\}$ have the same succession rule $(k) \rightsquigarrow (2)(3)\dots(k+1)$, but different succession functions :

- $T = \{312\}$, $\chi_T(i,k) = i + 1$



- $T = \{321\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ i & \text{otherwise} \end{cases}$



Colored regular pattern

- *color* : each permutation associated with an integer c [Barcucci, Pinzani, ...].
- if $\pi \in \mathfrak{S}_n(T)$ with color c, the insertion of $n+1$ in its i-th active site produces a $\sigma \in \mathfrak{S}_{n+1}(T)$ with $\mu(i, \pi, c)$ active sites and color $\nu(i, \pi, c)$;
- we extend the previous χ function in order to transform a triple $(i, k, c) \in \mathbb{N}^3$ into a couple $(\mu(i, k, c), \nu(i, k, c)) \in \mathbb{N}^2$.

Colored regular pattern

A set of patterns T is called c-regular if

- the length one $1 \in \mathfrak{S}_1(T)$ has two sons,
- all active sites are right justified,
- for any $n \geq 1$ and $\pi \in \mathfrak{S}_n(T)$, $\chi_T(i, \pi, c)$ does not depend on π but only on i, on c and on the number k of active sites of π. In this case we denote $\chi_T(i, \pi, c)$ by $\chi_T(i, k, c) = (\mu(i, k, c), \nu(i, k, c))$ and χ_T becomes a function $\chi_T : \mathbb{N}^3 \to \mathbb{N}^2$; it generalizes the succession function $\chi(i, k)$.

$$(k_c) \rightsquigarrow (\mu(1, k, c)_{\nu(1,k,c)})(\mu(2, k, c)_{\nu(2,k,c)}) \ldots (\mu(k, k, c)_{\nu(k,k,c)})$$

or $(k_c) \rightsquigarrow \cup_{i=1}^{k} (\mu(i, k, c)_{\nu(i,k,c)})$, for $k \geq 1$,

is called *colored succession rule* corresponding to the set T

Example : Even index Fibonacci numbers

The succession function $\chi : \mathbb{N}^3 \rightarrow \mathbb{N}^2$ of even index Fibonacci numbers corresponds to $T = \{312, 2431\}$ [Do, Vajnovszki 2007] is identified by $\chi(i, k, c) = (\mu(i, k, c), \nu(i, k, c))$ with :

$$\mu(i, k, c) = \begin{cases} i + 1 & \text{if } i = 1 \text{ or } (i = k \text{ and } c \neq 1) \\ i & \text{otherwise} \end{cases}$$

$$\nu(i, k, c) = \begin{cases} 0 & \text{if } i = 1 \text{ or } (i = k \text{ and } c \neq 1) \\ 1 & \text{otherwise} \end{cases}$$

Corresponding succession rule :

$$\begin{cases} (k_0) \rightsquigarrow (2_0)(2_1)\dots(k-1)_1(k+1)_0 \\ (k_1) \rightsquigarrow (2_0)(2_1)\dots(k_1) \end{cases}$$

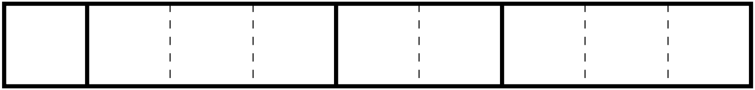
General Generating Algorithm [Do, Vajnovszki 2007]

First call : Gen_Avoid$(1, 2, 0)$

procedure Gen_Avoid$(size, k, c)$
local i, u, v
if $size = n$ **then**
  Print(π)
else
  $size := size + 1$
  $\pi := [\pi, size]$
  $(\mu, \nu) := \chi(1, k, c)$
  gen_Avoid$(size, \mu, \nu)$
  for $i := 2$ **to** k **do**
    $\pi := \pi \cdot (size - i + 2, size - i + 1)$
    $(u, v) := \chi(i, k, c)$
    gen_Avoid$(size, \mu, \nu)$
  end for
  for $i := k$ **downto** 2 **do**
    $\pi := \pi \cdot (size - i + 2, size - i + 1)$
  end for
end if
end procedure

Bijections

$1 + 3 + 2 + 3$



$0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad \cancel{0}$

$1 \quad 3 \quad 4 \quad 2 \quad 6 \quad 5 \quad 8 \quad 9 \quad 7$

Composition $\leftrightarrow$ Pavage $\leftrightarrow$ Binary string $\leftrightarrow$ Permutation

Compositions of n

$\leftrightarrow$ Binary strings of length $n-1$
$\leftrightarrow$ $\mathfrak{S}_n\{321, 312\}$, $\mathfrak{S}_n\{321, 231\}$

Compositions of n with all parts of sizes $\leq p$

$\leftrightarrow$ Binary strings of length $n-1$ without p consecutive 1s $\leftrightarrow$ $\mathfrak{S}_n\{321, 312, 234\dots(p+1)1\}$, $\mathfrak{S}_n\{321, 231, (p+1)p\dots321\}$
$\leftrightarrow$ p-generalized Fibonacci numbers [Baril, Do 2006]

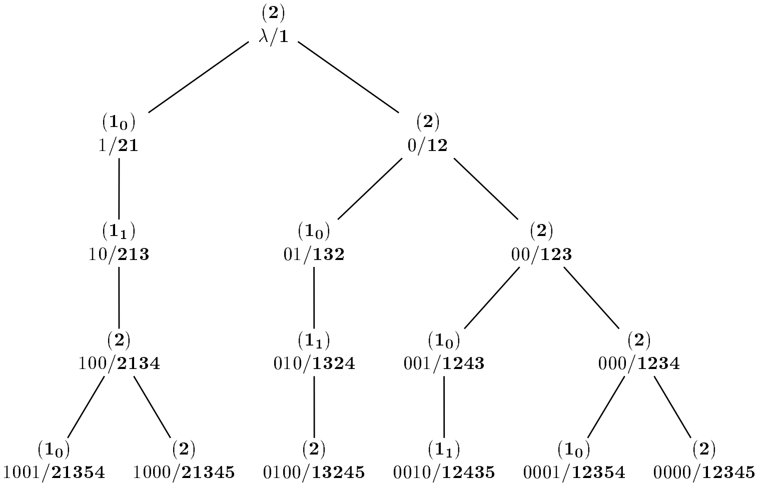
Theorem

A system of succession rules for $\mathcal{C}_{1,p}(n)$:

$$(\Omega_p)\left\{\begin{array}{ll}(2) & \\ (2) \rightsquigarrow (2)(1_0) & \\ (1_i) \rightsquigarrow (1_{i+1}), & \text{for } 0 \leq i < p-2 \\ (1_{p-2}) \rightsquigarrow (2). & \end{array}\right.$$

- *The generating tree (Ω_p) is coded by the permutations in $\mathfrak{S}_n(231, 312, 321, 21\overline{34\ldots(p+1)(p+3)}(p+2))$.*
- *This tree is also coded by the set $\mathcal{B}_{\geq p-1}(n)$ of binary strings of length n with at least $(p-1)$ zeros between two ones.*

$\mathcal{C}_{1,3}(n)$



The level n is coded by $\mathfrak{S}_n(231, 312, 321, 21\overline{34}65)$ or by $\mathcal{B}_{\geq 2}(n-1)$

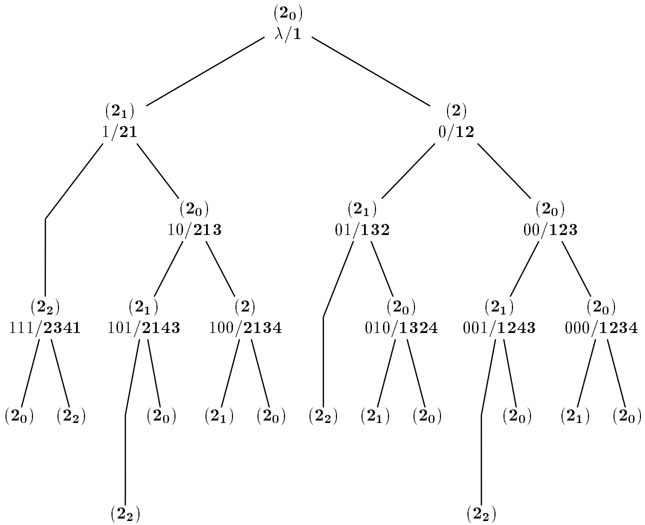
Theorem

A system of jumping succession rules (Φ_p) *for* $\mathcal{C}_{\hat{p}}(n)$ *:*

$$(\Phi_p)\left\{\begin{array}{lll} (2_0) & & \\ (2_i) \rightsquigarrow (2_0)(2_{i+1}), & & \text{for } 0 \le i \le p-2 \\ (2_{p-2}) \overset{1}{\rightsquigarrow} (2_0) & & \\ \qquad\quad \overset{2}{\rightsquigarrow} (2_{p-1}) & & \\ (2_{p-1}) \rightsquigarrow (2_0)(2_{p-1}). & & \end{array}\right.$$

- *The generating tree* (Φ_p) *is coded by the permutations in*
$$\mathfrak{S}_n(312, 321, T_p) \text{ with } T_p = \left\{\begin{array}{l} \bar{2}3\ldots(p+1)1 \\ 2\bar{3}\ldots(p+1)1 \\ \ldots \\ 23\ldots\overline{(p+1)}1. \end{array}\right.$$
- *This tree is also coded by the set* $\mathcal{B}_{\hat{p}-1}(n)$ *of binary strings of length* n *without runs of* 1*s of length* $(p-1)$.

$\mathcal{C}_{\hat{3}}(n)$



The level n is coded by $\mathcal{B}_{\hat{2}}(n-1)$ or by $\mathfrak{S}_n(312, 321, T_3)$ with $T_3 = \{\bar{2}341, 2\bar{3}41, 23\bar{4}1\}$

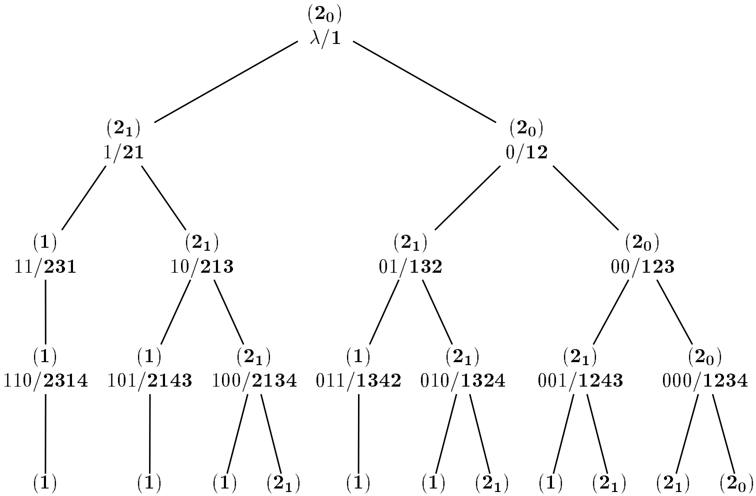
Theorem

A system of succession rules (Θ_p) for $\mathcal{C}_{\#p}(n)$ is given by :

$$(\Theta_p)\begin{cases}(2_0) & \\ (2_i) \rightsquigarrow (2_i)(2_{i+1}), & \text{for } 0 \le i < p-2 \\ (2_{p-2}) \rightsquigarrow (2_{p-2})(1) & \\ (1) \rightsquigarrow (1). & \end{cases}$$

- *The generating tree (Θ_p) is coded by the permutations in $\mathfrak{S}_n(312, 321, H_p)$ with H_p is defined by :*
  - *for $p = 2$, $H_2 = \{231, 2143\}$,*
  - *let $\tau = \tau(1)\tau(2)\dots\tau(k)$ be a pattern in H_{p-1} $(k \le 2(p-1))$. We identify : $H_p = \cup_{\tau \in H_{p-1}}\{\tau(1)\tau(2)\dots\tau(k-1)(k+1)\tau(k)$, $\tau(1)\tau(2)\dots\tau(k)(k+2)(k+1)\}$. H_p contains 2^{p-1} patterns.*
- *This tree is also coded by the set $\bar{\mathcal{B}}_{\#p-1}(n)$ of binary strings of length n having at most $(p-1)$ ones.*

$\mathcal{C}_{\#p}(n)$



The level n is coded by $\bar{\mathcal{B}}_{\#2}(n)$ or by $\mathfrak{S}_n(312, 321, H_3)$ with $H_3 = \{2341, 23154, 21453, 214365\}$

Summary

Classes	Succession rules	Avoidance patterns
$C(n)$	(2) $(2) \rightsquigarrow (2)(2)$	$\{321, 312\}$
	(2) $(k) \rightsquigarrow (k+1)(1)^{k-1}$	$\{321, 231\}$
$C_{\leq p}(n)$	(2_0) $(2_0) \rightsquigarrow (2_0)(2_1)$ $(2_i) \rightsquigarrow (2_0)(2_{i+1})$, $(2_{p-2}) \rightsquigarrow (2_0)(1)$ $(1) \rightsquigarrow (2_0)$	$\{321, 312, 234\ldots(p+1)1\}$
	(2) $(k) \rightsquigarrow (k+1)(1)^{k-1}$ $(p) \rightsquigarrow (p)(1)^{k-1}$	$\{312, 231, (p+1)p\ldots 321\}$
$C_{1,p}(n)$, $C(n+1, p, 1)$	(2) $(2) \rightsquigarrow (1_0)(2)$ $(1_i) \rightsquigarrow (1_{i+1})$, for $0 \leq i < p-1$ $(1_{p-1}) \rightsquigarrow (2)$	$\{231, 312, 321,$ $2134\ldots(p+1)(p+3)(p+2)\}$
$C_{\hat{p}}(n)$	(2_0) $(2_i) \rightsquigarrow (2_0)(2_{i+1})$, for $0 \leq i < p-2$ $(2_{p-1}) \overset{1}{\rightsquigarrow} (2_0)$ $\overset{2}{\rightsquigarrow} (2)$ $(2) \rightsquigarrow (2_0)(2)$	$\{312, 321, T_p\}$, where $T_p = \begin{cases} \bar{2}3\ldots(p+1)1 \\ 2\bar{3}\ldots(p+1)1 \\ \ldots \\ 23\ldots\overline{(p+1)}\,1 \end{cases}$
$C_{\#p}(n)$	(2_0) $(2_i) \rightsquigarrow (2_i)(2_{i+1})$, for $0 \leq i \leq p-2$ $(2_{p-2}) \rightsquigarrow (2_{p-2})(1)$ $(1) \rightsquigarrow (1)$	$\{312, 321, H_p\}$
$C_*(n, p, r)$	(2_0) $(2_0) \overset{1}{\rightsquigarrow} (2)$ $\overset{p}{\rightsquigarrow} (2_0)$ $(2) \rightsquigarrow (2)(2)$	for $C_*(n, 3, r)$: $\{312, 4321, 2431, 3241, 3421,$ $3\overline{21}654, \overline{32}1654, \bar{3}2\bar{1}654\}$

Algorithmic consideration

- An algorithm is Constant Amortized Time (**CAT**) if the number of computations after a small amount of preprocessing is proportional to the number of objects generated.
- Almost all classes of pattern avoiding permutations found here are regular.
- Establish the corresponding succession functions from these succession rules.
- Apply the general generating algorithms in order to efficiently generate the permutations corresponding to these studied compositions.

CAT requirements

[Ruskey,Vajnovszki 2002]

If a recursive generating procedure satisfies the following properties :

- the amount of computation of a given call is proportional to its degree, disregarding the recursive calls,
- each call has the degree zero or at least two, and
- at the completion of each recursive call a new word is generated,

then the generating procedure is CAT.

Almost all succession rules here induce generating trees whose nodes have at least two successors. This situation satisfies the requirements of a CAT algorithm.