Wreath-closed permutation classes

Mike Atkinson, Nik Ruškuc, and Rebecca Smith

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Definition

The *basis* of a pattern class is the maximum set of minimal permutations that are avoided by all permutations in the pattern class.

Note that the set of permutations avoiding a particular permutation or set of permutations is a closed class.

The permutation $\alpha[\beta_1, \ldots, \beta_n]$ is such that the ith term of α is substituted by β_i. In other words $\alpha[\beta_1, \ldots, \beta_n]$ consists of n segments order isomorphic to $\beta_1, \ldots, \beta_n$ where the relative order of the segments is the same as the relative order of the terms of α.

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Example

Let $\alpha = 2413, \beta_1 = 123, \beta_2 = 21, \beta_3 = 1, \beta = 312$. Then $\alpha[\beta_1, \ldots, \beta_4] = 234\ 98\ 1\ 756$.

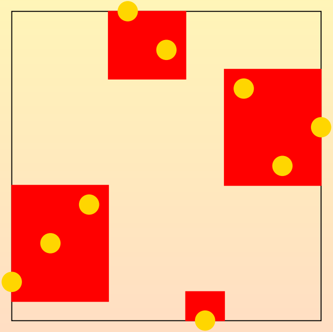

Figure: $\alpha[\beta_1, \ldots, \beta_4] = 234\ 98\ 1\ 756$.

Definition

A pattern class X is said to be wreath-closed (or substitution closed) if $\alpha[\beta_1, \ldots, \beta_n] \in X$ for all $\alpha, \beta_1, \ldots, \beta_n \in X$.

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In this talk, we consider the wreath-closure of $X = Av(\psi)$ where ψ is any permutation. In particular, does the wreath-closure of X have a finite or infinite basis?

Definition

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Example

The permutation 58147362 is simple.

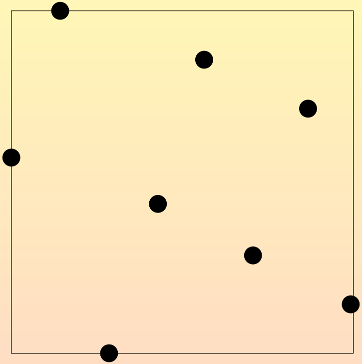

Figure: The permutation 58147362.

In their article "Simple permutations and pattern restricted permutations", Albert and Atkinson show the crucial connection between wreath-closed classes and simple permutations:

Proposition

A pattern class is wreath-closed if and only if its basis consists of simple permutations.

Definition

An extension ξ of ψ is a *minimal simple extension* of ψ if

1. ξ is simple, and
2. among all simple extensions of ψ, ξ is minimal under the subpermutation order.

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Lemma

The basis of the wreath closure of $X = Av(\psi)$ is the set of minimal simple extensions of ψ.

Example

A case where the wreath-closure of $X = Av(\psi)$ has an infinite basis.

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Example

A case where the wreath-closure of $X = Av(\psi)$ has an infinite basis.

Let $\psi = 1234$.

Then the basis of the wreath-closure of X contains the set: $\{35861472,$
$57(10)8361492, \ldots,$
$(2k+1)(2k+3)(2k+6)(2k+4)(2k-1)(2k+2)(2k-3)(2k+4)\cdots583614(2k+5)2, \ldots\}$

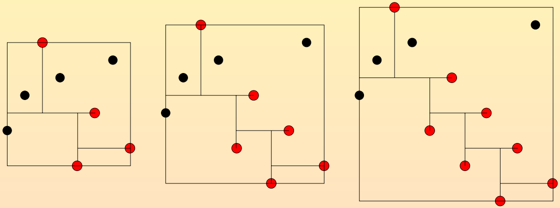

Figure: Three basis elements of $AV(1234)$.

To define pin sequences, we consider the graphs of permutations.

Example

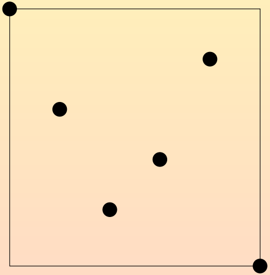

Figure: The graph of the permutation 642351.

The rectangle of a set of points of the graph of permutation is the (minimum) rectangle that contains these points.

Example

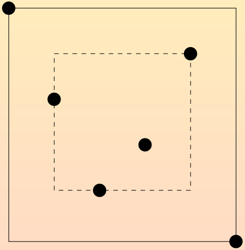

Figure: Rect(4,2,3,5).

Proper pin sequences were first defined by Brignall, Huczynska, and Vatter in their paper "Decomposing simple permutations, with enumerative consequences."

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Definition

If p_1, p_2 are two points of a permutation then a *proper pin sequence* from $\{p_1, p_2\}$ is a sequences of points $p_1, p_2, p_3, \ldots$ such that, for each $i \geq 2$,

1. p_{i+1} lies outside $\mathrm{Rect}(p_1, p_2, \ldots, p_i)$,
2. p_{i+1} cuts $\mathrm{Rect}(p_1, p_2, \ldots, p_i)$ either to the left, right, below or above it,
3. p_{i+1} is extremal in its direction with respect to $\mathrm{Rect}(p_1, p_2, \ldots, p_i)$,
4. p_{i+1} separates p_i from $\mathrm{Rect}(p_1, p_2, \ldots, p_{i-1})$ by lying vertically or horizontally between p_i and $\mathrm{Rect}(p_1, p_2, \ldots, p_{i-1})$.

We also rely on the following two propositions of Brignall, Huczynska, and Vatter:

Proposition

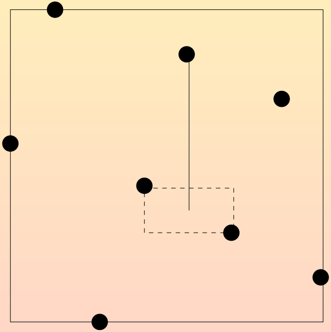
If $P = p_1, p_2, \ldots, p_m$ is a pin sequence in a permutation σ then the only subsets of P that can be non-trivial intervals are $\{p_1, p_m\}$, $\{p_2, p_m\}$, $\{p_1, p_3, \ldots, p_m\}$, and $\{p_2, p_3, \ldots, p_m\}$.

Proposition

If p_1, p_2 are points of a simple permutation σ then there is a pin sequence $P = p_1, p_2, \ldots, p_m$ whose final point is the last point of σ (a right-reaching pin sequence). Similarly, there is a left-reaching pin sequence whose first two points are p_1, p_2.

Example

A pin sequence extending from $p_1 = 4$ and $p_2 = 3$ in the simple permutation 58147362



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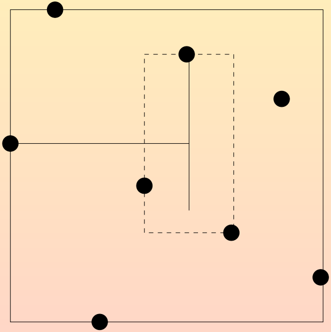
Example

A pin sequence extending from $p_1 =$ and p_2 in the simple permutation 58147362



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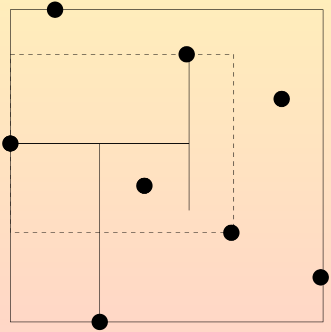

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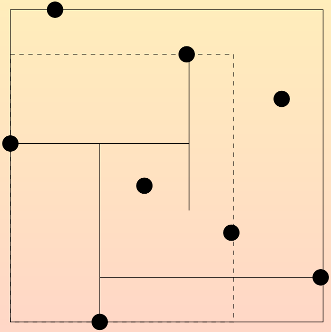

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Lemma

Let α be any permutation except for $1, 12, 21, 132, 213, 231, 312$. Then there are arbitrarily long pin sequences not containing α as a subsequence.

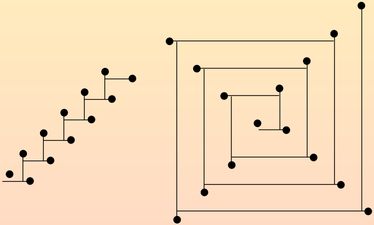


Figure: Pin sequences P_1, P_2.

Lemma

Let $P = p_1, p_2, \ldots$ be any pin sequence of length 5 or more and let $p_a, p_{a+1}, p_{a+2}, p_{a+3}, p_{a+4}$ be five consecutive points of P. Let θ be any of $12, 21, 132, 213, 231, 312$. Then among $p_a, p_{a+1}, p_{a+2}, p_{a+3}, p_{a+4}$ we can find a permutation isomorphic to θ and two points r, s in this permutation such that $r, s, p_{a+5}, p_{a+6}, \ldots$ is a pin sequence.

Lemma

Let $P = p_1, p_2, \ldots$ be a pin sequence that avoids 2413. Then starting no later than p_4, the steps of P will be repetitions of the pattern BRAL (or a cyclic variant). Similarly, if P avoids 3142, then the steps will repeat the pattern LARB (or a cyclic variant).

Lemma

Consider the diagrams shown below where the minimal intervals of θ and ϕ are separated by hook points. Suppose that either $\theta \neq \tau \ominus 1$ or $\phi \neq 1 \ominus \omega$. This permutation is simple and, unless $\theta = \iota_s$ and $\psi = \iota_t$, the permutation $\theta \ominus \phi$ embeds uniquely in it.

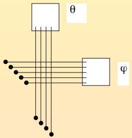


Figure: Simple extension of $\theta \ominus \phi$

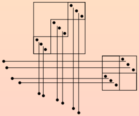


Figure: Simple extension of $\theta \ominus \phi$ when $\theta = \oplus_{i=1}^{s} \delta_r$ and $\phi = \oplus_{i=1}^{t} \delta_r$.

Lemma

Consider the diagrams shown in below where the minimal intervals of $\theta = \tau \ominus 1$ and $\phi = 1 \ominus \omega$ are separated by the hook points and θ and ϕ are separated by a hook point whose position is based on the relative sizes of θ and ϕ. This permutation is simple and, unless $\theta = \phi = \iota_1$, the permutation $\theta \ominus \phi$ embeds uniquely in it.

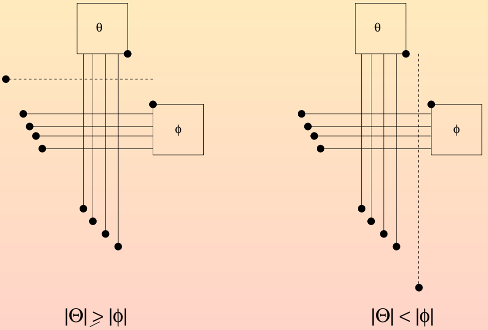


Figure: Simple extension of $\theta \ominus \phi$ where $\theta = \tau \ominus 1$ and $\phi = 1 \ominus \omega$.

Proposition

Let α be any permutation. Then there exists a simple permutation χ such that $\alpha \preceq \chi$ and α is the unique copy of α within χ.

If the simple skeleton is 12 or 21 use the constructions of the previous lemmas or:

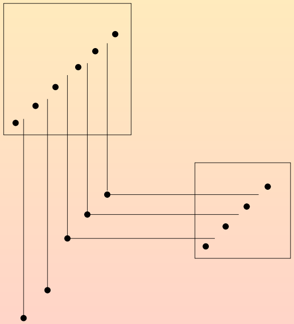
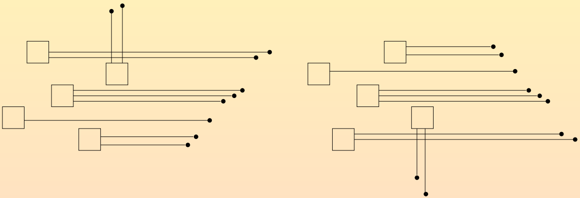


Figure: A simple extension of $\iota_s \ominus \iota_t$.

If the length of the simple skeleton is at least 4, use hooks:



θ_{n-1} is not the top interval $\qquad$ θ_{n-1} is the top interval

Figure: Simple extension of $\sigma[\theta_1, \ldots, \theta_n]$

Lemma

Let α be a permutation of the form 123, 321, or of length at least 4. Then there exists a simple permutation α_s^* that contains α, minimal subject to its simplicity. Furthermore there are permutations $\hat{\alpha}$ of arbitrary length which consist of α_s^* and a pin sequence P such that

1. *$\hat{\alpha} = \alpha_s^* \cup P$ is simple and contains a unique copy of α, and*
2. *If $\alpha \preceq \beta \preceq \hat{\alpha}$ and β is simple then β has the form $\alpha_s^* \cup P_0$ where P_0 is an initial subsequence of the pin sequence P.*

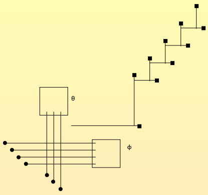


Figure: The permutation $\hat{\alpha}$ for $\alpha = \theta \ominus \phi \neq \iota_s \ominus \iota_t$.

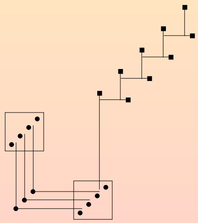


Figure: The permutation $\hat{\alpha}$ for $\alpha = \iota_s \ominus \iota_t$.

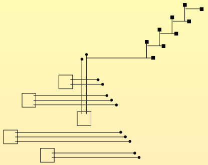


Figure: The permutation $\hat{\alpha}$ for $\alpha = \sigma[\theta_1, \ldots, \theta_n]$ where $n \geq 4$ and $\alpha \neq 3142, 2413$.

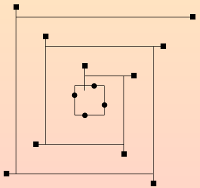


Figure: The permutation $\hat{\alpha}$ for $\alpha = 3142$

Theorem

Suppose that the permutation $\psi = \sigma[\alpha_1, \ldots, \alpha_n]$ where σ is the simple skeleton of ψ and $|\sigma| = n \geq 4$. If each α_i is a subpermutation of 132, 213, 231 or 312, then ψ has finite type.

Theorem

If ψ is of any of the following types:

1. $231 \oplus 231$, $312 \oplus 312$, $231 \oplus 312$, $312 \oplus 231$ *or any subpermutation of these,*
2. $21 \oplus 1 \oplus 21$, $1 \oplus 21 \oplus 1$, *or any subpermutation of these, or*
3. $2413 \oplus 1$, $1 \oplus 2413$, $3142 \oplus 1$, *or* $1 \oplus 3142$,

then ψ has finite type.

Definition

A *spiral permutation* ψ is made up of a centre pattern 3142 and a contiguous sequence of pin points from the clockwise pin sequence $p_i p_{i+1} \ldots p_k$ where $i \geq 2$ as shown below. Furthermore, $|\psi| > 4$. Also a *(dual) spiral permutation* ψ is made up of a centre pattern 2413 and a contiguous sequence of pin points from the counter-clockwise pin sequence $p_i p_{i+1} \ldots p_k$ where $i \geq 2$.

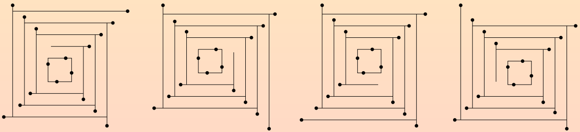


Figure: Spiral permutations with centre permutation 3142.

Lemma

Suppose a permutation ψ is contained in a spiral permutation ζ, its simple decomposition has more than one interval, and its only interval of size greater than two is a 3142 [or 2413] pattern. Then ψ is itself a spiral permutation.

Lemma

A spiral permutation ψ where $|\psi| \neq 6, 7, 9$ has only one non-trivial interval, that is, the centre 3142 [or 2413] pattern. If $|\psi| = 9$, then ψ has exactly two non-trivial intervals where the second of these intervals is made up of the first and last pin points. If $|\psi| = 6, 7$, then ψ has a plus or minus decomposition with one interval of length 1 and another interval of length $|\psi| - 1$.

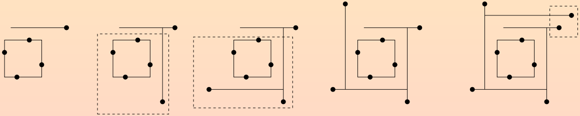


Figure: The permutation ψ when $5 \leq |\psi| \leq 9$.

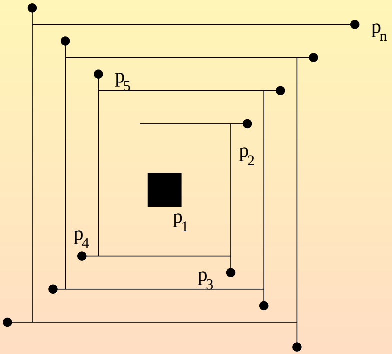


Figure: A spiral permutation with centre permutation 3142.

Theorem
Any spiral permutation ψ where $|\psi| \neq 6, 7$ has finite type.

Theorem

Suppose ψ is plus decomposable and is not of any the form already stated to be of finite type, nor any of its symmetries. Then ψ has infinite type.

Some cases shown here:

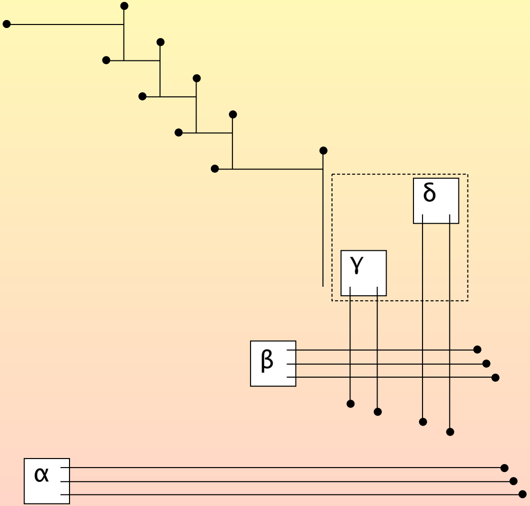


Figure: $\alpha \oplus \beta \oplus \gamma \oplus \delta$ has infinite type.

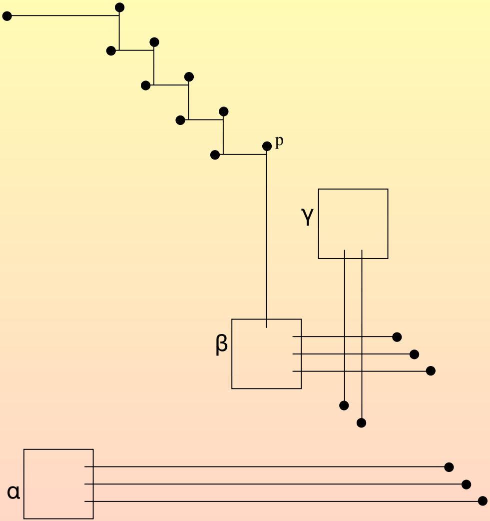


Figure: $\alpha \oplus \beta \oplus \gamma$ has infinite type when not of the forms: $21 \oplus 1 \oplus 21$, $1 \oplus 21 \oplus 1$, or any subpermutation of these.

Consider a permutation with simple decomposition

$$\psi = \sigma[\alpha_1, \alpha_2, \ldots, \alpha_n]$$

with $n \geq 4$ that is not a spiral.

Additionally, assume that at least one of the intervals α_i is isomorphic to 123, or isomorphic to 321, or has length at least four.

Choose a special interval α_s as:

1. If any of $\alpha_1, \ldots, \alpha_n$ are of length 4 and not isomorphic to 2413 or 3142 then we choose α_s to be the longest of such intervals.
2. Else, if there are no intervals of length greater than 4 then we choose α_s to be any interval isomorphic 2413 or 3142.
3. Else, if there are no intervals of size 4 or more we choose α_s to be any 123 or 321 interval.

All the symmetries of ψ are of the same type (all finite, or all infinite) and so we may replace ψ by any of its symmetries. If $\alpha_s = a_1 \cdots a_k$ then, by replacing ψ by its reverse if necessary, we can assume that $a_k < a_1$. Furthermore we may assume that $s < n$ since, if $s = n$, we can pass to the reverse complement of ψ (and have $s = 1$).

Let α_s^* be a minimal simple permutation containing α_s. Let $P = \{p_1, \ldots, p_m\}$ be an arbitrarily long pin sequence out of α_s^*. Do this so that $\alpha_s^* \cup P$ is simple and has a unique subpermutation isomorphic to α_s.

We form the permutation shown like those in the following figures. (Some variations occur in some extraordinary cases.)

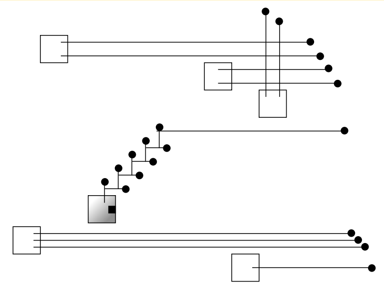


Figure: The permutation ξ if $\alpha_s \neq 3142$.

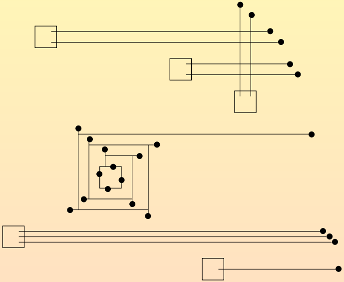


Figure: The permutation ξ if $\alpha_s = 3142$.

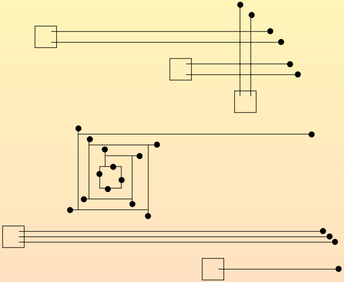


Figure: The permutation ξ if $\alpha_s = 3142$.

The constructions (or the necessary variants thereof) will prove to be minimal simple extensions of ψ and thus show that ψ is of infinite type.

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Table: Bases for finite wreath closures of small permutation classes

ψ	Basis of the wreath closure of $Av(\psi)$
231	2413,3142
123	24153, 25314, 31524, 41352, 246135, 415263
3142	3142
3412	35142, 42513, 351624, 415263, 246135
4132	41352, 35142, 263514, 531642, 264153, 526413, 362514
4231	463152, 364152, 264153, 536142, 531642, 531462 462513, 362514, 263514, 526413, 524613, 524163 526314, 426315, 513642, 362415, 461352, 416352
4312	463152, 364152, 264153, 536142, 531642, 531462 462513, 362514, 263514, 526413, 524613, 524163 526314, 426315, 513642, 362415, 461352, 416352

Table: Classification of $\psi = \sigma[\alpha_1, \ldots, \alpha_n]$ when $n \leq 2$

α_1	α_2	$\sigma = 1$	$\sigma = 12$	$\sigma = 21$
1	$\emptyset$	F	—	—
1	1	—	F	F
1	$\{21, 231, 312\}$	—	F	∞
$\{21, 231, 312\}$	1	—	F	∞
1	$\{12, 132, 213\}$	—	∞	F
$\{12, 132, 213\}$	1	—	∞	F
$\{21, 231, 312\}$	$\{21, 231, 312\}$	—	F	∞
$\{12, 132, 213\}$	$\{12, 132, 213\}$	—	∞	F
$\{2413, 3142\}$	1	—	F	F
1	$\{2413, 3142\}$	—	F	F
All other combinations		—	∞	∞

Table: Classification of $\psi = \sigma[\alpha_1, \ldots, \alpha_n]$ when $n = 3$

α_1	α_2	α_3	$\sigma = 123$	$\sigma = 321$
1	1	1	F	F
21	1	1	F	∞
1	1	21	F	∞
21	1	21	F	∞
1	21	1	F	∞
12	1	1	∞	F
1	1	12	∞	F
12	1	12	∞	F
1	12	1	∞	F
All other combinations			∞	∞

Table: Classification of $\psi = \sigma[\alpha_1, \ldots, \alpha_n]$ when $n \geq 4$, ψ is not a spiral.

α_i	$\sigma = 12\ldots n$ $\sigma = n\ldots 21$	σ simple
All $\alpha_i \in \{1, 12, 21, 132, 231, 213, 312\}$	∞	F
Some $\alpha_i \notin \{1, 12, 21, 132, 231, 213, 312\}$	∞	∞

Table: Classification of Spiral Permutations ψ.

k	ψ
5	F
6,7	∞
8,9,10,...	F

Thank you!