Avoiding consecutive patterns in permutations

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January 3, 2009

Abstract

The number of permutations that do not contain, as a factor (subword), a given set of permutations Π is studied. A new treatment of the case $\Pi = \{12 \cdots k\}$ is given and then some numerical data is presented for sets Π consisting of permutations of length at most 4. Some large sets of Wilf-equivalent permutations are also given.

1 Introduction

The notion of one permutation π being contained in another permutation σ generally refers to σ having a subsequence π' that is order isomorphic to π. In *generalised* pattern containment extra conditions are stipulated relating to when terms of π' should be adjacent in σ. The extreme case of this, which we call *consecutive pattern containment*, is when we require all the terms of π' to be consecutive:– so π is consecutively contained in σ if σ has a factor that is order isomorphic to π. For example 521643 contains 132 because of the factor 164.

A frequently studied problem in pattern containment is to enumerate the permutations of each length that fail to contain (or *avoid*) one or more given patterns. In the case of consecutive pattern containment the first systematic study of avoidance problems was by Elizalde and Noy [4]. To describe their results and to present our own contribution we introduce some notation.

Let Π be a set of permutations. Define $\text{Cav}(\Pi)$ as the set of all permutations σ that (consecutively) avoid every permutation in Π. If Π consists of a single permutation π we write this as $\text{Cav}(\pi)$.

To enumerate sets of this form it turns out that the exponential generating function

$$\sum_n t_n x^n / n!$$

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(where t_n is the number of permutations in the set of length n) is the natural tool and Elizalde and Noy used it to obtain results on avoiding the increasing pattern $\iota_k = 12\cdots k$ (and less complete results for other patterns). Their approach was to consider the more general problem of counting permutations with a specific number of occurrences of ι_k, define appropriate bivariate generating functions, set up differential equations satisfied by these functions, and solve these equations (at least in principle). In particular they found the explicit generating functions for the sets Cav(123) and Cav(1234) as

$$\frac{\sqrt{3}}{2}\frac{\exp(y/2)}{\cos(\sqrt{3}y/2+\pi/6)}$$

and

$$\frac{2}{\cos y - \sin y + \exp(-y)}$$

respectively.

In section 2 we shall give a new approach to enumerating Cav(ι_k). Our method also requires bivariate generating functions and (partial) differential equations but it enumerates the permutations according to the value of their final term.

Elizalde and Noy [4] also began the study of sets Cav(Π) where Π consists of permutations of length 3. This was continued by Kitaev and Mansour [6, 7]. This work resulted in fairly complete enumeration results with the exceptions of the two sets Cav$\{312, 132\}$ and Cav$\{312, 231\}$. For the latter two sets we give enumerating recurrences in section 3. We have used the recurrences to get approximations to certain limits and it is convenient here to contrast such limits with the corresponding situation for classical pattern containment.

For classical pattern containment one defines Av(Π) to be the set of permutations that classically avoid all the permutations of a given set Π. Given such a set Π let p_n be the number of permutations in Av(Π)) of length n. Then the Marcus-Tardos theorem [8] is that, if Π is non-empty, $p_n \leq k^n$ for some k that depends only on Π. It is also known [2] that, when Π is a singleton set, $\lim \sqrt[n]{p_n}$ exists but it is a tantalising open problem to prove this in general.

Elizalde [3] observed that, for consecutive pattern containment, the situation is similar. In particular he proved that, if c_n is the number of permutations of length n in Cav(σ), then

$$\frac{c_n}{n!} \leq k^n$$

for some constant $k < 1$, and

$$\lim \sqrt[n]{\frac{c_n}{n!}}$$

exists. It is obvious that the first of these extends to any non-empty set Π of avoided permutations. However, the second remains true also as is easily confirmed by checking Elizalde's proof. It is this type of limit that we approximate in Section 3. In this section we also present some computational data for sets Cav(σ) with $|\sigma| = 4$.

Of the sets that are defined by avoiding permutations of length 3 two are particularly noteworthy. The first is the set Cav$\{123, 321\}$ which consists of permutations which alternate in rises and descents. They are clearly related to the alternating permutations of André [1] and their generating function is $2(\sec x + \tan x) - x - 1$. The second is the set Cav$\{123, 231, 312\}$ which in [7] is shown to have the generating function $1 + x(\sec x + \tan x)$. In section 4 we give a one-to-one correspondence between two subsets of these sets that explains the similarity in their generating functions.

Just as for classical pattern avoidance we can define two sets Π and Π' to be *Wilf-equivalent* if the enumerations of Cav(Π) and Cav(Π') are equal. A byproduct of Theorem 3.2 of [4] was to prove that the singleton sets $\{12 \cdots a\ \tau a + 1\}$ (where τ is any permutation of $a + 2, a + 3, \ldots$) are Wilf-equivalent. In the last section we give a substantial generalisation of this result.

2 Avoiding $123 \cdots k$

In this section we consider permutations that avoid $\iota_k = 12 \cdots k$. Define

$$U_{na}^{(t)}$$

to be the set of permutations π such that

1. π avoids ι_k,
2. π has length $n \geq 0$,
3. the final term term of π is a where $1 \leq a \leq n$, and
4. π ends in t rises where, necessarily, $t \leq k - 2$

Put $u_{na}^{(t)} = |U_{na}^{(t)}|$.

Permutations of $U_{na}^{(t)}$ that have $t = 0$ (so end in a descent) arise by appending a final element a to a permutation $\sigma \in U_{n-1,b}^{(s)}$ where $b \geq a$ and $0 \leq s \leq k-2$. It follows that

$$u_{na}^{(0)} = \sum_{b:b \geq a} \left(u_{n-1,b}^{(0)} + u_{n-1,b}^{(1)} + \cdots + u_{n-1,b}^{(k-2)} \right)$$

On the other hand permutations of $U_{na}^{(t)}$ that have $t > 0$ arise by appending a final element a to a permutation $\sigma \in U_{n-1,b}^{(t-1)}$ where $b \leq a$. This gives

$$u_{na}^{(t)} = \sum_{b:b<a} u_{n-1,b}^{(t-1)}$$

with $t > 0$.

We can rewrite these two equations without the summations over b as

$$\begin{aligned} u_{na}^{(0)} &= u_{n,a+1}^{(0)} + u_{n-1,a}^{(0)} + u_{n-1,a}^{(1)} + \cdots + u_{n-1,a}^{(k-2)} \\ u_{na}^{(t)} &= u_{n,a-1}^{(t)} + u_{n-1,a-1}^{(t-1)} \text{ if } t > 0 \end{aligned}$$

where appropriate initial conditions are

$$\begin{aligned} u_{11}^{(0)} &= 1 \\ u_{nn}^{(0)} &= 0 \text{ if } n \geq 1 \\ u_{n1}^{(t)} &= 0 \text{ if } t > 0 \end{aligned}$$

We now make a change of variables and put

$$v_{ij}^{(t)} = u_{i+j+1,i+1}^{(t)}$$

With these variables the equations and initial conditions become

$$\begin{aligned} v_{00}^{(0)} &= 1 \\ v_{i0}^{(0)} &= 0 \text{ if } i \geq 0 \\ v_{0j}^{(t)} &= 0 \text{ if } t > 0 \\ v_{ij}^{(0)} &= v_{i+1,j-1}^{(0)} + v_{i,j-1}^{(0)} + v_{i,j-1}^{(1)} + \cdots + v_{i,j-1}^{(k-2)} \\ v_{ij}^{(t)} &= v_{i-1,j+1}^{(t)} + v_{i-1,j}^{(t)} \text{ where } t > 0 \end{aligned}$$

Next we introduce the (exponential) generating functions

$$V^{(t)}(x,y) = \sum_{i,j} v_{ij}^{(t)} \frac{x^i}{i!} \frac{y^j}{j!}$$

and rewrite the recurrences as equations between the generating functions. This gives the differential equations

$$\begin{aligned} \frac{\partial V^{(0)}}{\partial y} &= \frac{\partial V^{(0)}}{\partial x} + V^{(0)} + V^{(1)} + \cdots V^{(k-2)} \\ \frac{\partial V^{(1)}}{\partial x} &= \frac{\partial V^{(1)}}{\partial y} + V^{(0)} \\ \frac{\partial V^{(2)}}{\partial x} &= \frac{\partial V^{(2)}}{\partial y} + V^{(1)} \\ &\cdots \\ \frac{\partial V^{(k-2)}}{\partial x} &= \frac{\partial V^{(k-2)}}{\partial y} + V^{(k-3)} \end{aligned}$$

and the boundary value equations are

$$\begin{aligned} V^{(0)}(x,0) &= 1 \\ V^{(t)}(0,y) &= 0 \text{ for } 1 \le t \le k-2 \end{aligned}$$

To solve these equations we substitute $w = (x+y)/2, z = (x-y)/2$. The chain rule for differentiation tells us that

$$\begin{aligned} \frac{\partial V^{(t)}}{\partial x} &= \frac{1}{2}\left(\frac{\partial V^{(t)}}{\partial w} + \frac{\partial V^{(t)}}{\partial z}\right) \\ \frac{\partial V^{(t)}}{\partial y} &= \frac{1}{2}\left(\frac{\partial V^{(t)}}{\partial w} - \frac{\partial V^{(t)}}{\partial z}\right) \end{aligned}$$

so we can reformulate the equations as:

$$\begin{aligned} \frac{\partial V^{(0)}}{\partial z} &= -\left(V^{(0)} + V^{(1)} + \cdots V^{(k-2)}\right) \\ \frac{\partial V^{(1)}}{\partial z} &= V^{(0)} \\ \frac{\partial V^{(2)}}{\partial z} &= V^{(1)} \\ & \ldots \\ \frac{\partial V^{(k-2)}}{\partial z} &= V^{(k-3)} \end{aligned}$$

Now, by elimination, we get a $(k-1)$th order differential equation for $V^{(k-2)}$

$$\sum_{i=0}^{k-1} \frac{\partial^i V^{(k-2)}}{\partial z^i} = 0$$

The solution of this equation is

$$V^{(k-2)} = A_1 \exp(\lambda_1 z) + A_2 \exp(\lambda_2 z) + \cdots + A_{k-1} \exp(\lambda_{k-1} z)$$

where $\lambda_1, \ldots, \lambda_{k-1}$ are the roots of

$$\lambda^{k-1} + \lambda^{k-2} + \cdots + 1 = 0$$

which are the non-trivial kth roots of 1. Here $A_1, \ldots, A_{k-1}$ are independent of z, and so are functions of w alone. The other variables $V^{(k-3)}, V^{(k-4)}, \ldots, V^{(0)}$ may be found by successive differentiation with respect to z.

To find $A_1, \ldots, A_{k-1}$ we use the boundary conditions. These show that $A_1, \ldots, A_{k-1}$ satisfy the following equations:

$$
\begin{aligned}
A_1 \exp(-\lambda_1 w) + A_2 \exp(-\lambda_2 w) + \cdots + A_{k-1} \exp(-\lambda_{k-1} w) &= 0 \\
A_1 \lambda_1 \exp(-\lambda_1 w) + A_2 \lambda_2 \exp(-\lambda_2 w) + \cdots + A_{k-1} \lambda_{k-1} \exp(-\lambda_{k-1} w) &= 0 \\
\cdots & \\
A_1 \lambda_1^{k-3} \exp(-\lambda_1 w) + A_2 \lambda_2^{k-3} \exp(-\lambda_2 w) + \cdots + A_{k-1} \lambda_{k-1}^{k-3} \exp(-\lambda_{k-1} w) &= 0 \\
A_1 \lambda_1^{k-2} \exp(\lambda_1 w) + A_2 \lambda_2^{k-2} \exp(\lambda_2 w) + \cdots + A_{k-1} \lambda_{k-1}^{k-2} \exp(\lambda_{k-1} w) &= 1
\end{aligned}
$$

To solve these equations we regard the first $k - 2$ of them as homogeneous linear equations for the quantities $A_i \exp(-\lambda_i w)$ and solve the system whose matrix is

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{k-1} \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{k-1}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_1^{k-3} & \lambda_2^{k-3} & \lambda_3^{k-3} & \cdots & \lambda_{k-1}^{k-3}
\end{bmatrix}
$$

by the method of determinants. Let Δ_i denote the value of the determinant of the above matrix when the ith column is removed, multiplied by a sign ($+1$ if i is odd, -1 if i is even). Because the determinant is of van der Monde type its value is, to within sign,

$$
\prod_{r<s, r\neq i, s\neq i} (\lambda_r - \lambda_s)
$$

Then we have

$$
\frac{A_1 \exp(-\lambda_1 w)}{\Delta_1} = \frac{A_2 \exp(-\lambda_2 w)}{\Delta_2} = \cdots = \frac{A_{k-1} \exp(-\lambda_{k-1} w)}{\Delta_{k-1}}
$$

If the value of this ratio is denoted by R we can find R by substituting into the final equation. This gives (noting that $\lambda_i^{k-2} = \lambda_i^{-2}$)

$$
R = \frac{1}{\lambda_1^{-2} \exp(2\lambda_1 w)\Delta_1 + \lambda_2^{-2} \exp(2\lambda_2 w)\Delta_2 + \cdots + \lambda_{k-1}^{-2} \exp(2\lambda_{k-1} w)\Delta_{k-1}}
$$

Hence

$$
A_i = \frac{\exp(\lambda_i w)\Delta_i}{R}
$$

Therefore, using $w = (x+y)/2$ and $w + z = x$, the generating function $V^{(k-2)}(x, y)$ is

$$
\frac{\exp(\lambda_1 x)\Delta_1 + \exp(\lambda_2 x)\Delta_2 + \cdots + \exp(\lambda_{k-1} x)\Delta_{k-1}}{\lambda_1^{-2} \exp(\lambda_1 (x+y))\Delta_1 + \lambda_2^{-2} \exp(\lambda_2 (x+y))\Delta_2 + \cdots + \lambda_{k-1}^{-2} \exp(\lambda_{k-1} (x+y))\Delta_{k-1}}
$$

The expressions for $V^{(k-3)}, \ldots, V^{(0)}$ are similar in that the denominators are the same but the numerators have multiplying coefficients that are powers of the λ_i.

Examples of these functions are

1. $n = 3$
$$V^{(0)}(x, y) = \frac{\exp(y/2)\cos(\sqrt{3}x/2 + \pi/6)}{\cos(\sqrt{3}(x + y)/2 + \pi/6)}$$

2. $n = 4$
$$V^{(0)}(x, y) = \frac{\cos x - \sin x + \exp(-x)}{\cos(x + y) - \sin(x + y) + \exp(-x - y)}$$

Now observe that $v^{(0)}_{0n} = u^{(0)}_{n+1,1}$ is by definition the number of permutations of length $n + 1$ that end with the term 1. This however is the same as the total number of permutations in the set of length n. In particular, the generating function that enumerates $\mathrm{Cav}(\iota_k)$ is $V^{(0)}(0, y)$ and this is (again using $\lambda_i^{k-2} = \lambda_i^{-2}$)

$$\frac{\lambda_1^{-2}\Delta_1 + \lambda_2^{-2}\Delta_2 + \cdots + \lambda_{k-1}^{-2}\Delta_{k-1}}{\lambda_1^{-2}\exp(\lambda_1 y)\Delta_1 + \lambda_2^{-2}\exp(\lambda_2 y)\Delta_2 + \cdots + \lambda_{k-1}^{-2}\exp(\lambda_{k-1} y)\Delta_{k-1}}$$

Two instances of the use of this formula are

1. The generating function of $\mathrm{Cav}(123456)$ is
$$\frac{3}{\exp(x/2)\cos(z + \pi/3) + \sqrt{3}\exp(-x/2)\cos(z + \pi/6) + \exp(-x)}$$
where $z = \sqrt{3}x/2$

2. The generating function of $\mathrm{Cav}(12345678)$ is
$$\frac{4}{\exp(-x) + \cos(x) - \sin(x) + 2\cos(z)\cosh(z) - \sqrt{2}\cos(z)\sinh(z) - \sqrt{2}\cosh(z)\sin(z)}$$
where $z = \sqrt{2}x/2$.

We conclude this section with an observation. The one variable generating function $V^{(0)}(0, y)$ which enumerates the permutations that avoid ι_k is obtained by putting $x = 0$ in the two variable generating function $V^{(0)}(x, y)$. But, in fact, $V^{(0)}(x, y)$ can be obtained also from $V^{(0)}(0, y)$. The reason is that, from the formulae above, $V^{(0)}(x, y)$ is a quotient of the form
$$V^{(0)}(x, y) = \frac{h(x)}{h(x + y)}$$
and therefore
$$V^{(0)}(0, y) = \frac{h(0)}{h(y)}$$
from which we obtain
$$V^{(0)}(x, y) = \frac{h(x)}{h(x + y)} = \frac{\big(h(0)/V^{(0)}(0, x)\big)}{\big(h(0)/V^{(0)}(0, x + y)\big)} = \frac{V^{(0)}(0, x + y)}{V^{(0)}(0, x)}$$

3 Recurrences and limits

In this section we find approximations for various limits of the form $\lim \sqrt[n]{t_n/n!}$ where t_n is the number of permutations in various pattern avoidance sets. This is done by finding recurrence relations that allow us to compute t_n for values of n far beyond what could be achieved by generating the permutations in the set directly. The method we use has appeared in the pattern literature before but the only general description of it that we have been able to find is the brief hint in the final section of [5]. We then compute the limit by calculating

$$\frac{t_n}{nt_{n-1}}$$

for large values of n (which has faster convergence behaviour than computing $\lim \sqrt[n]{t_n/n!}$ directly).

We begin by considering Cav$(312, 132)$ and Cav$(312, 231)$. In both cases we enumerate the permutations according to the values of their final two terms a, b and we distinguish $a < b$ from $a > b$. We then go on to consider Cav(σ) with $|\sigma| = 4$. Our results indicate that there are no unexpected equalities in the values of these limits.

3.1 Avoiding 312 and 132

Let u_{nab} be the number of permutations of length n in Cav$(312, 132)$ that end with a rising ab and d_{nab} the number that end with a descending ab. A permutation of length n in Cav$(312, 132)$ that ends with ab arises by appending b to one of length $n-1$ (with appropriate renumbering of terms).

If $a < b$ we have to ensure that the final three terms cab of this new permutation are not isomorphic to 312. If $c < a$ this is already assured but if $c > a$ we need also $c < b$. This gives

$$u_{nab} = \sum_{c:c<a} u_{n-1,c,a} + \sum_{c:a<c<b} d_{n-1,c,a}$$

If $a > b$ we need to ensure that the final three terms are not isomorphic to 132. Taking into account the necessary renumbering of terms we obtain

$$d_{nab} = \sum_{c:b<c<a} u_{n-1,c-1,a-1} + \sum_{c:c>a} d_{n-1,c-1,a-1}$$

The total number t_n of avoiders of length n is then

$$\sum_{a,b} u_{nab} + \sum_{a,b} d_{nab}$$

We have used these recurrences to compute over 100 terms of the sequence (t_n). The first 10 terms are given in Table 1. We have the approximation

$$\lim_{n\to\infty} \sqrt[n]{\frac{t_n}{n!}} = 0.601730727943943$$

n	1	2	3	4	5	6	7	8	9	10
t_n	1	2	4	10	30	108	454	2186	11840	71254

Table 1: $\mathrm{Cav}(312, 132)$

n	1	2	3	4	5	6	7	8	9	10
t_n	1	2	4	11	37	149	705	3814	23199	156940

Table 2: $\mathrm{Cav}(312, 231)$

3.2 Avoiding 312 and 231

Again we define u_{nab} to be the number of such permutations that end with a rising ab and d_{nab} the number that end with a descending ab. Arguing as before leads to

$$\begin{aligned} u_{nab} &= \sum_{c:c<a<b} u_{n-1,c,a} + \sum_{c:a<c<b} d_{n-1,c,a} \\ d_{nab} &= \sum_{c:c<b} u_{n-1,c-1,a-1} + \sum_{c:c>a-1} d_{n-1,c,a-1} \end{aligned}$$

The total number t_n of avoiders of length n is then

$$\sum_{a,b} u_{nab} + \sum_{a,b} d_{nab}$$

Here the approximate limit is

$$\lim_{n\to\infty} \sqrt[n]{\frac{t_n}{n!}} = 0.676388228094035$$

and the first 10 terms are given in Table 2.

3.3 Avoiding patterns of length 4

For the enumeration of a set of the form $\mathrm{Cav}(\sigma)$ with $|\sigma| = 4$ it is enough, as explained in [4], to consider 7 permutations only, namely 1234, 2413, 2143, 1324, 1423, 1342, and 1243. In the case of 1234, 1342, and 1243 the enumerations can be read off from the generating functions found in [4]. For the other four permutations we have computed the number of permutations of length n in $\mathrm{Cav}(\sigma)$ for $n \leq 34$ using the same method as above but keeping track of the final three terms of the permutations rather than the final two. We give the data for $n \leq 10$ in Table 3 and, in Table 4, give the limits we have found (and, for comparison purposes, also the limits given in [4]). Elizalde

n	1	2	3	4	5	6	7	8	9	10
2413	1	2	6	23	110	632	4237	32465	279828	2679950
2143	1	2	6	23	110	631	4223	32301	277962	2657797
1324	1	2	6	23	110	632	4229	32337	278204	2659223
1423	1	2	6	23	110	631	4218	32221	276896	2643883

Table 3: Cav(σ) with $|\sigma| = 4$

σ	$\lim \sqrt[n]{t_n}$
1234	0.963005
2413	0.957718
2143	0.956174
1324	0.955850
1423	0.954826
1342	0.954611
1243	0.952891

Table 4: Approximate limits

and Noy noted an unusual comparison between the enumerations of Cav(1324) and Cav(2143): the latter is smaller than the former for $n \leq 11$ but greater for $n = 12$. For $n \leq 34$ we have found no further oddities.

4 Cav$(123, 321)$ and Cav$(123, 231, 312)$

In their paper [7] Kitaev and Mansour compute the generating function for the set Cav$(123, 231, 312)$ as $1 + x(\sec x + \tan x)$ which, of course, is strikingly similar to the generating function for Cav$(123, 321)$, namely, $2(\sec x + \tan x) - x - 1$. It follows quite readily from their analysis that the numbers of permutations that start with the symbol 1 are the same for each set. In this section we give an explicit bijection between these two sets of permutations by producing an infinite tree that describes both sets.

Let A be the set of permutations in Cav$(123, 321)$ that start with the symbol 1 and B the set of permutations in Cav$(123, 231, 312)$ that also start with the symbol 1. For both A and B we construct an infinite tree whose nodes are labelled by permutations in the set, the root being labelled by the permutation 1 and the nodes at level n being labelled by permutations of length n. In this tree the parent of a (non-root) node is obtained by removing the final symbol (and renumbering the terms). The top fragment of both trees is shown in Figure 1 and they have been drawn to suggest that they may be isomorphic.

Lemma 1 *In the tree for A the children of a node at level n with down degree d have down degrees $n - d + 1, n - d + 2, \ldots, n$.*

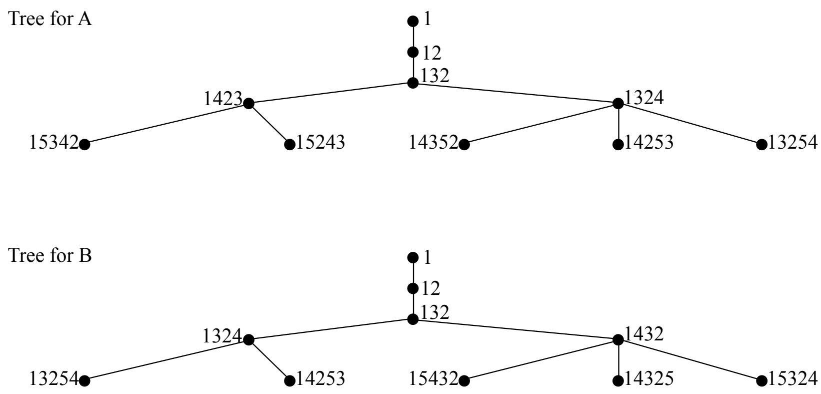

Figure 1: Trees for the sets A and B

Proof: The permutations of A are alternating and begin with 1. If a permutation $\pi \in A$ has even length it must end with an ascent and so, if its final symbol is k, its children in the tree are obtained from π by appending one of the symbols $2, 3, \ldots, k$ (with appropriate renumbering of prior terms); in particular it has $k-1$ children. On the other hand a permutation $\pi \in A$ of odd length n must end with a descent and, if its final symbol is k, its children are obtained by appending one of $k+1, k+2, \ldots, n+1$ to π; such a node has $n+1-k$ children.

Consider a node π at level n with d children. If n is even π must end with $d+1$ and its children must end with one of $2, 3, \ldots, d+1$. However these children are at odd depth $n+1$ and so their down degrees are (respectively) $n+2-2, n+2-3, \ldots, n+2-(d+1)$, in other words $n, n-1, \ldots, n-d+1$. If n is odd then π must end with $n-d+1$ and have children ending with one of $n-d+2, n-d+3, \ldots, n+1$. These children are at even depth $n+1$ and so their down degrees are $n-d+1, n-d+2, \ldots, n$ also. ■

Lemma 2 *In the tree for B the children of a node at level n with down degree d have down degrees $n-d+1, n-d+2, \ldots, n$.*

Proof: Consider a node π of the tree for B that has depth n and suppose that it ends with the two terms ℓ, k. If $\ell < k$ its children arise by appending terms with values $\ell+1, \ell+2, \ldots, k$ (with appropriate renumbering of previous terms); thus it has $k-\ell$ children and they end with the final pairs $(k+1, \ell+1), (k+1, \ell+2), \ldots, (k+1, k)$. On the other hand if $\ell > k$ the children arise from appending terms with values $2, 3, \ldots, k$ and $\ell+1, \ell+2, \ldots, n+1$; the node therefore has $n+k-\ell$ children and their final pairs are $(k+1, 2), (k+1, 3), \ldots, (k+1, k)$ and $(k, \ell+1), (k, \ell+2), \ldots, (k, n+1)$.

Now consider a node π at level n with d children and suppose that its final two terms are ℓ, k. Suppose first that $\ell < k$. Then $d = k - \ell$, the children of π have final pairs

$(k+1,\ell+1),(k+1,\ell+2),\ldots,(k+1,k)$ and they have down degrees

$$n+1+(\ell+1)-(k+1),n+1+(\ell+2)-(k+1),\ldots,n+1+k-(k+1)$$

or $n-d+1,n-d+2,\ldots,n$.

Next suppose that $\ell>k$. Here $d=n+k-\ell$ and the children of π end in the final pairs $(k+1,2),(k+1,3),\ldots,(k+1,k)$ and $(k,\ell+1),(k,\ell+2),\ldots,(k,n+1)$. Their down degrees are therefore

$$n+1+2-(k+1),n+1+3-(k+1),\ldots,n+1+k-(k+1)$$

and

$$\ell+1-k,\ell+2-k,\ldots,n+1-k$$

so the set of down degrees is again $\{n-d+1,n-d+2,\ldots,n\}$. ∎

These two lemmas suffice to show that the trees for the two sets are isomorphic and so we have a bijection between the two sets.

5 Equal distributions

In this section we give some families of Wilf-equivalent sets as a consequence of a more general result on equal distributions.

If α and β are permutations then $\alpha\oplus\beta$ (and, respectively, $\alpha\ominus\beta$) denotes the permutation which is the concatenation of words α' isomorphic to α and β' isomorphic to β with $\alpha'<\beta'$ (respectively, $\alpha'>\beta'$).

A pair of permutations (α,β) is said to have the *separation* property if $1\ominus\beta$ is not (consecutively) contained in α and $\alpha\oplus 1$ is not (consecutively) contained in β.

Let $\alpha\beta$ be any permutation and let $\Pi(\alpha,\beta,k)$ be the set of all permutations $\alpha\gamma\beta$ where $|\gamma|=k$. Notice that this notation implies that the terms of γ are greater than the terms of $\alpha\beta$.

Lemma 3 *Let $\alpha\beta$ be any permutation such that the pair of permutations isomorphic to α,β has the separation property. Let σ be any permutation and let $\alpha_1\gamma_1\beta_1,\ldots,\alpha_t\gamma_t\beta_t$ be the factors of σ that are isomorphic to permutations of $\Pi(\alpha,\beta,k)$. Then, for all $i\neq j$, $\alpha_i\gamma_i\beta_i\cap\gamma_j$ is empty.*

Proof: Suppose that, for some $i\neq j$, $\alpha_i\gamma_i\beta_i\cap\gamma_j$ is not empty. Since $\gamma_i\neq\gamma_j$ the initial symbol of γ_i precedes or succeeds the initial symbol of γ_j. Suppose the former possibility occurs. Then $\beta_i\cap\gamma_j$ is not empty and contains a term q. If there was term $p\in\alpha_j\cap\gamma_i$ we would have both $p<q$ since $\alpha_j<\gamma_j$ and $p>q$ since $\gamma_i>\beta_i$; hence $\alpha_j\cap\gamma_i$ is empty. It follows that β_i contains the whole of α_j and since it contains also at least the first symbol of γ_j (which exceeds α_j) it contains $\alpha\oplus 1$. This contradicts the separation property.

A similar argument proves that if the initial symbol of γ_i succeeds the initial symbol of γ_j the separation property is again violated through $1\ominus\beta$ being contained in α. ∎

Theorem 1 *Suppose that $\alpha\beta$ is a permutation such that the pair of permutations isomorphic to α, β has the separation property. Let*

$$\Theta = \{\theta_1, \ldots, \theta_t\} \subseteq \Pi(\alpha, \beta, k)$$

for some k, where $\theta_i = \alpha_i\gamma_i\beta_i$, and let $m_1, \ldots, m_t$ be any non-negative integers. Then the number of permutations of length n that contain θ_i exactly m_i times depends only on $\alpha, \beta, k, t, m_1, \ldots, m_t$ and not on the permutations γ_i themselves.

Proof: We define an equivalence relation $\sim$ on the permutations of length n as follows. If σ and σ' are of length n let $\alpha_1\delta_1\beta_1, \ldots, \alpha_u\delta_u\beta_u$ be the sequence of factors of σ that are isomorphic to permutations in $\Pi(\alpha, \beta, k)$ and let $\alpha'_1\delta'_1\beta'_1, \ldots, \alpha'_u\delta'_u\beta'_v$ be the corresponding sequence for σ'. Then $\sigma \sim \sigma'$ if

1. $u = v$
2. α_i and α'_i occur at the same positions within σ and σ' and have corresponding equal values,
3. β_i and β'_i occur at the same positions within σ and σ' and have corresponding equal values,
4. δ_i and δ'_i have the same set of values (though not necessarily in the same order),
5. The sequence of terms of σ not among $\alpha_1\delta_1\beta_1, \ldots, \alpha_u\delta_u\beta_u$ is equal to the the sequence of terms of σ' not among $\alpha'_1\delta'_1\beta'_1, \ldots, \alpha'_u\delta'_u\beta'_v$

The permutations in any given equivalence class are characterised by the class itself and by the order of terms within each factor δ_i. By Lemma 3 each δ_i can be arranged in any order. In order that a permutation of the equivalence class contains each θ_i exactly m_i times it is necessary and sufficient that among its factors $\delta_1, \ldots, \delta_u$ there are m_i of them isomorphic to γ_i. But the number of ways in which this can happen depends on $k, t, m_1, \ldots, m_t, u$ only. ∎

Corollary 1 *Suppose that $\alpha\beta$ is a permutation such that the pair of permutations isomorphic to α, β has the separation property and suppose that k and t are fixed. Then all t-element subsets of $\Pi(\alpha, \beta, k)$ are Wilf-equivalent.*

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