Cyclically closed pattern classes of permutations

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Abstract

The cyclic closure of a permutation pattern class is defined as the set of all the cyclic rotations of its permutations. Examples of finitely based classes whose cyclic closure is also finitely based are given, as well as an example where the cyclic closure is not finitely based. Some enumerations of cyclic closures are computed.

1 Introduction

We shall investigate the connection between two relations on permutations: pattern containment and cyclic rotation.

A permutation π is said to be a *subpermutation* of (or pattern within) a permutation σ if σ has a subsequence that is order isomorphic to π. Within the last few years this notion has been studied intensively, see [5] and references therein. The subpermutation relation is generally studied by means of *closed classes* of permutations: those that are closed downwards under forming subpermutations. Such classes can always be specified by a set of avoided permutations, and the minimal avoided set is called the *basis* of the class. The class defined by an avoided set B is denoted by $\mathrm{Av}(B)$.

Typical questions about closed classes are:

1. How many permutations of each length are contained in the class?

2. What is the basis of the class; in particular, is it finite?

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Cyclic rotation is a much simpler relation: σ is a *cyclic rotation* of π if there is some reading of π, regarded as a circular list, that agrees with σ. Clearly, every permutation of length n is related by cyclic rotation to exactly n permutations (including itself).

To study how these two relation interact we define *cyclically closed* classes of permutations: these are classes closed downwards under taking subpermutations, and closed also under taking cyclic rotations.

If we are given the basis of a closed class it is trivial to verify whether the class is cyclically closed: we just have to check that the basis is closed under all cyclic rotations. That observation leads to a simple way to construct the largest cyclically closed class contained in some given closed class: we simply take the basis of the given class, form all the cyclic rotations of its permutations, discard the ones not minimal under taking subpermutations, and take the result as the basis of a new class. In particular, if the original basis was finite, so also will be the basis of the new class.

This suggests a dual problem: given a closed class X, find the smallest cyclically closed class containing it (we call this the *cyclic closure* of X and denote it by $\mathrm{cc}(X)$). Notice that $\mathrm{cc}(X)$ can be obtained from X by forming all the rotations of the permutations of X. We shall be interested in how to enumerate $\mathrm{cc}(X)$ given an enumeration of X, and how its basis is related to the basis of X.

In section 2 we shall develop some techniques to describe the basis of a cyclic closure and we shall use them to prove that the cyclic closures of classes of the form $\mathrm{Av}(\sigma)$ are finitely based whenever σ has length 3. By contrast we shall give an example of a permutation σ of length 6 where the cyclic closure of $\mathrm{Av}(\sigma)$ has an infinite basis.

The enumerations (s_n) and (t_n) of a closed class X and its cyclic closure $\mathrm{cc}(X)$ clearly satisfy

$$s_n \leq t_n \leq ns_n$$

and, from this, one readily deduces that, if $\lambda = \lim_{n\to\infty} \sqrt[n]{s_n}$ exists, then so does $\lim_{n\to\infty} \sqrt[n]{t_n}$ and this is also equal to λ. To get more precise enumeration information it is necessary to work out which cyclic rotations of permutations in X also lie in X. We shall give some example enumerations in section 3.

2 Pattern restrictions

In this section we shall study the basis of a class $\mathrm{cc}(X)$ where $X = \mathrm{Av}(B)$ itself is given by its basis B. Since the permutations of both $\mathrm{cc}(X)$ and its complement are closed under cyclic rotation we shall often consider them to be circular lists; in particular, when we refer to either a subsequence or a segment we shall allow wrap-around subsequences and segments. For example, $312, 261$ are subsequences of 361524, and $152, 2436$ are segments.

The basis of $\mathrm{cc}(X)$ is the set of permutations minimal with respect to not lying in $\mathrm{cc}(X)$. Therefore we begin by giving a condition that a permutation does not lie in $\mathrm{cc}(X)$; in other words, none of its cyclic rotations lie in X. This requires two definitions.

Let σ be any permutation. A subsequence θ of σ that is order isomorphic to a permutation of B is called a *basic subsequence*; although basic subsequences are defined with respect to the basis B, we shall omit a notational reference to B since it will always be clear from the context. Given a basic subsequence θ with first term s and last term t let $\tilde{\theta} = s\beta\gamma t$ be the segment that begins with s and ends with t, and write

$$\sigma = \alpha s\beta\gamma t\delta \text{ or } \sigma = \gamma t\delta\alpha s\beta$$

according to whether $\tilde{\theta} = s\beta\gamma t$ wraps or not. The rotations of σ that begin with a term of the segment $W(\theta) = \delta\alpha s$ all contain θ as a subsequence without wrap-around; so none of these permutations lie in X. We say that θ *witnesses* the points of the segment $W(\theta)$, or that $W(\theta)$ is a *witnessed segment*.

Lemma 1 *$\sigma \notin cc(X)$ if and only if the witnessed segments cover σ.*

Proof: The remarks above already prove that, if the witnessed segments of σ cover σ, then no cyclic rotation of σ lies in X; so $\sigma \notin \mathrm{cc}(X)$. Conversely, if there is some term that lies in no witnessed segment then the cyclic rotation of σ that begins with this term contains no basic subsequence (without wrap-around) and so lies in X; thus $\sigma \in \mathrm{cc}(X)$. ∎

Proposition 2 *Let $X = Av(B)$, where B is finite and suppose there is a bound Δ depending on B alone such that, for all $\sigma \notin cc(X)$, there is a collection of at most Δ witnessed segments that cover σ. Then $cc(X)$ is finitely based.*

Proof: Suppose that $\sigma \notin \mathrm{cc}(X)$. By hypothesis there is a set $\theta_1, \ldots, \theta_d$ with $d \leq \Delta$ for which the witnessed segments $W(\theta_1), \ldots, W(\theta_d)$ cover σ.

Consider the subsequence σ' of σ that consists of the points of all of $\theta_1, \ldots, \theta_d$ and renumber its points so that it is a permutation. Clearly, the witnessed segments $W(\theta_1), \ldots, W(\theta_d)$ of σ' cover σ'; so, by the previous lemma, $\sigma' \notin \mathrm{cc}(X)$. Furthermore, σ' is a union of at most Δ segments, each of length no longer than M, where M is the maximal length of a permutation of B; thus $|\sigma'| \leq M\Delta$.

We have proved that every permutation not in $\mathrm{cc}(X)$ has a subpermutation also not in $\mathrm{cc}(X)$ of bounded length. This proves that there are only finitely many permutations minimal with respect to not lying in $\mathrm{cc}(X)$; thus $\mathrm{cc}(X)$ is finitely based. ∎

The following theorems exploit this proposition.

Theorem 3 *$cc(Av(321))$ is finitely based.*

Proof: Suppose that $\sigma \notin cc(\mathrm{Av}(321))$ and write it as

$$\sigma = 1\alpha n\beta$$

after some suitable rotation. Suppose first that β is not increasing in which case we can find a descent vu of β; say $\beta = \gamma vu\delta$. The basic subsequences nvu and $vu1$ determine witnessed segments $\delta 1\alpha n$ and $\alpha n\gamma v$. Together these cover every point of σ except for u. However u is covered by some other witnessed segment and so, in this case, σ is covered by 3 witnessed segments.

Next suppose that $\beta = b_1 \cdots b_f$ is increasing and non-empty. By Proposition 2 the term b_1 lies in some witnessed segment $W(\theta)$ determined by some basic subsequence θ. Either θ has its largest term within β (in which case it may be taken to be b_f, with its next two terms $a_2, a_1 \in \alpha$) or its 3 decreasing terms a_3, a_2, a_1 all lie in α. In both cases $W(\theta)$ contains $b_1, \ldots, b_{f-1}$. However, nb_f1 is also a basic subsequence and this defines the witnessed segment αn. So, with the possible exception of b_f, every term of σ lies in one of two witnessed segments, and hence again σ can be covered by 3 witnessed segments.

Finally, suppose that β is empty. Then α cannot be increasing since $\sigma \notin \mathrm{Av}(321)$ and so it has the form $\alpha = \gamma dc\delta$ where dc is a descent. But now the basic subsequences ndc and $dc1$ define witnessed segments δn and γd. These cover every point of σ except for 1 and c, and hence σ can be covered by no more than 4 witnessed segments.

We can now apply Proposition 2 with $\Delta = 4$. ∎

Theorem 4 *$cc(Av(231))$ is finitely based.*

Proof: Suppose that $\sigma \notin \mathrm{cc}(\mathrm{Av}(231))$. We consider the longest chain of descents that end with the symbol 1 and the symbol u immediately preceding this chain. By taking a suitable cyclic rotation we can write

$$\sigma = uv_1v_2 \ldots v_rv_{r+1} \ldots v_{r+s}\beta$$

where $v_{r+s} = 1$ and

$$v_1 > v_2 > \ldots > v_r > u > v_{r+1} > \ldots > v_{r+s}.$$

We have a basic subsequence $\phi = uv_1v_{r+1}$ that witnesses the segment

$$W(\phi) = v_{r+2} \ldots v_{r+s}\beta u.$$

The term v_{r+1} lies in some other witnessed segment $W(\theta)$ say. Hence there is some basic subsequence $\theta = bca$ order isomorphic to 231 that begins at v_{r+1} or later and ends before v_{r+1}.

If a is not one of $v_1, v_2, \ldots, v_r$ then $W(\theta)$ contains all of these symbols. Then every point of σ except possibly v_{r+1} lies in $W(\theta) \cup W(\phi)$ and so σ can be covered by 3 witnessed segments.

On the other hand, if a is one of the symbols of $v_1, v_2, \ldots, v_r$, we may assume that c is also one of these symbols (if not, we may redefine a as u and revert to the previous case). Now it follows that $b \in \beta$ and c and a can be taken as some v_i and v_{i+1}; here $W(\theta)$ contains at least the points $v_{i+2}, \ldots, v_{r+1}$. However, as $b > v_{i+1}$, $v_{i+1}bu$ is a basic subsequence and its witnessed segment is $v_1, v_2, \ldots, v_{i+1}$. Again σ is covered by 3 witnessed segments and we can now apply Proposition 2. ■

The upper bounds on the length of basis elements of both cc(Av(321)) and cc(Av(231)) provided by Proposition 2 are excessive. In fact, computer calculations show these bases are, respectively,

1. $\{15432, 14325, 164253, 163254, 1472536\}$, and
2. $\{13425, 13524, 14253, 14523\}$

together with all their cyclic rotations.

We have succeeded in proving that cc(Av(σ)) is finitely based only for one permutation σ of length greater than 3.

Theorem 5 *cc(Av(4321)) is finitely based.*

Proof: Let $\sigma \notin$ cc(Av(4321)). Among the basic subsequences of σ choose one, $\theta = dcba$ say, spanning the smallest number of points. After a suitable rotation, write

$$\sigma = \delta d \gamma c \beta b \alpha a$$

Note that $W(\theta)$ is the interval δd. We want to find a bounded number of witnessed intervals that cover $\gamma c \beta b \alpha a$.

Figure 1 shows some of the structure of σ. Those blocks denoted by 0 are empty, and rising lines denote increasing subsequences; all these properties follow from the minimality of θ.

We shall begin by finding a single witnessed interval that contains the set Ξ of points in $c\beta b$ that are also greater than or equal to c (as Figure 1 shows, these points are increasing). Consider some witnessed segment $W(\phi)$ containing c. If the basic sequence ϕ does not begin in $c\beta b$ then $W(\phi)$ itself contains Ξ. On the other hand, if ϕ contains just one point of $c\beta b$, then that point is b or a point smaller than b (in which case it may be taken to be b itself) or it is a point of Ξ (in which case it may be taken to be the last point of Ξ); in all cases $W(\phi)$ contains the whole of Ξ as required.

By exploiting the reverse-complement symmetry we may, in the same way, find a single witnessed segment that contains all the points of $c\beta b$ that are smaller than or equal to b.

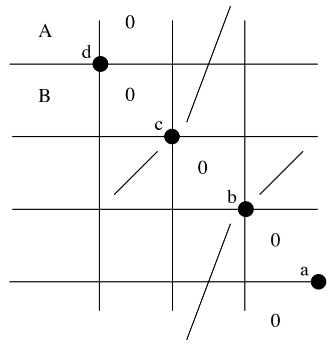

Figure 1: A basis element of cc(Av(4321))

We turn now to points of αa and find a single witnessed segment that contains all of them. If region A contains a point a' then $W(a'dcb)$ is such a segment; and if region B contains a descent b_2b_1 then $W(b_2b_1cb)$ is such a segment; so assume that neither of these occur. Consider some witnessed interval $W(\eta)$ that contains a. If $\eta = zyxw$ does not end in the segment α then $W(\eta)$ will contain αa as required; so assume that η ends within α; hence $b < w$. However, then η must begin within δ or the minimal property of $dcba$ will be contradicted. The second point y of η must also lie within δ or $dyxw$ would have smaller span than $dcba$; and hence, by the increasing property of B, y must be less than c. But it is now impossible that $x \in \alpha$; hence η may be replaced by $zyxb$ and now $\alpha a \in W(\eta)$.

Finally, using the reverse-complement symmetry again, all the points of $d\gamma$ lie in a single witnessed segment. ∎

Despite these examples, not every finitely based class has a finitely based cyclic closure:

Theorem 6 *The cyclic closure of Av*(265143) *is not finitely based.*

This theorem is proved by exhibiting infinitely many permutations $\pi_2, \pi_3, \ldots$ in the basis of Av(265143). The permutation π_n, where $n \geq 2$, is defined as the concatenation of segments:

$$\pi_n = A_n B_n C_n D_n E_n F_n G_n H_n I_n.$$

The segments themselves are defined below. In these definitions we separate the terms in a segment from one another by spaces and vertical bars; the types of separators have no significance other than to display the segment clearly.

$$
\begin{aligned}
A_n &= 4n+6\\
B_n &= 4n+10 \mid 4n+14 \mid 4n+13 \mid 4n+9 \mid 4n+12\\
C_n &= 4n+3\\
D_n &= 4n-1 \mid 4n-5 \mid 4n-9 \mid \cdots \mid 3\\
E_n &= 4n+11\\
F_n &= 2\,9\,5\,1\,4 \mid J_2\,J_3\,\cdots\,J_n\\
G_n &= 4n+8 \mid 4n+2\\
H_n &= 4n+4\\
I_n &= 4n+7
\end{aligned}
$$

where $J_i = 4i+5 \mid 4i-2 \mid 4i$.

The first two of these permutations are

$$\pi_2 = 14\,18\,22\,21\,17\,20\,11\,7\,3\,19\,2\,9\,5\,1\,4\,13\,6\,8\,16\,10\,12\,15$$

and

$$\pi_3 = 18\,22\,26\,25\,21\,24\,15\,11\,7\,3\,23\,2\,9\,5\,1\,4\,13\,6\,8\,17\,10\,12\,20\,14\,16\,19.$$

To further clarify the form of π_n we display the graph of π_{10} (a permutation on 54 points) in Figure 2.

Lemma 7 $\pi_n \notin cc(Av(2\,6\,5\,1\,4\,3))$.

Proof: We shall show that each term of π_n lies in a witnessed segment and then appeal to Proposition 2. First, observe that B_nE_n, $C_nE_nG_nI_nA_n$ and 295143 are all basic subsequences of π_n. We have that $F_nG_nH_nI_nA_n$ is contained in $W(B_nE_n)$, B_nC_n is contained in $W(C_nE_nG_nI_nA_n)$, and E_n is contained in $W(2\,9\,5\,1\,4\,3)$.

It remains to show that each term of D_n belongs to some witnessed segment. The terms $4i-1$ with $2 \le i \le n$ (that is, all terms of D_n except the final term) lie in witnessed segments of the form

$$W(4i-1 \mid 4n+11 \mid 4i+5 \mid 4i-2 \mid 4i+4 \mid 4i+3).$$

The final term 3 lies in

$$W(3 \mid 4n+11 \mid 9\ 1\ 8\ 7).$$

■

Lemma 8 *Every proper subpermutation of π_n belongs to $S = cc(Av(265143))$, for all $n \ge 2$.*

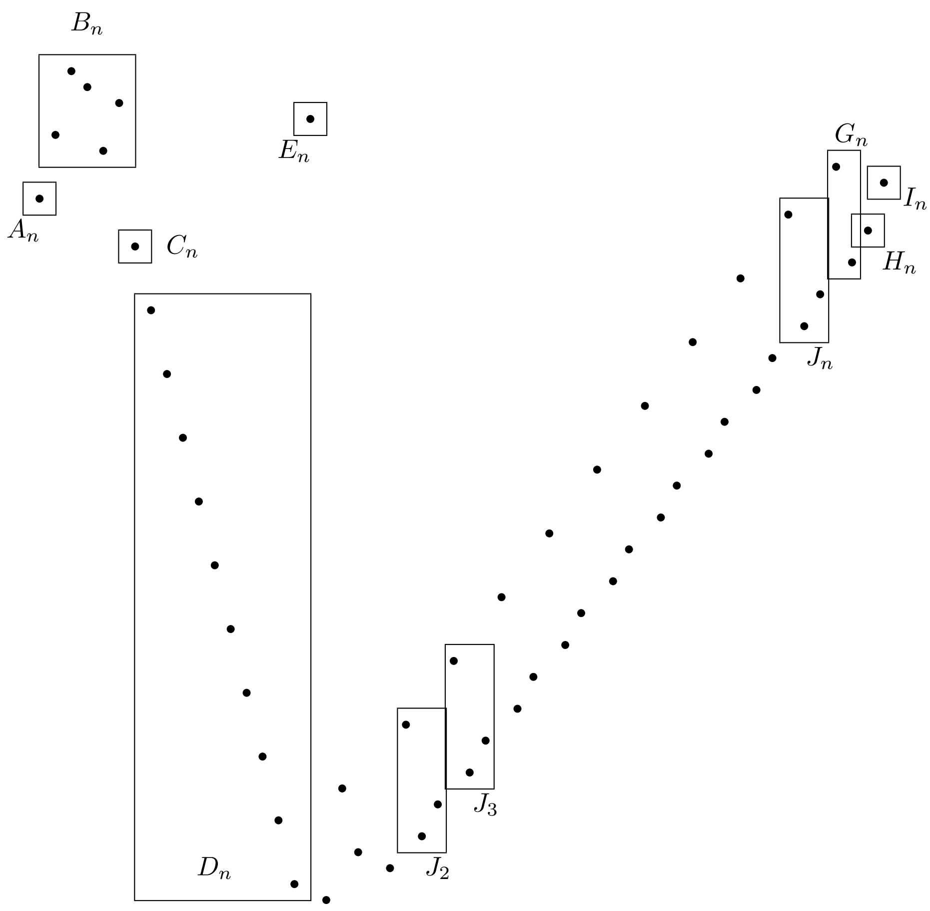

Figure 2: The graph of π_{10}, and its segments

Proof: We shall find a set of pairs $\mathcal{P} = \{(\sigma, p)\}$ with the following properties:

1. σ is a basic subsequence and p is a term of π_n,
2. $p \in W(\sigma)$ but p lies in no other witnessed segment.

Then we shall check that, for every term t of π_n, there exists a pair $(\sigma, p) \in \mathcal{P}$ such that t is a term of σ but $t \neq p$. From this we can deduce that, if t is removed from π_n, the term p of the result lies in no witnessed segment; therefore, by Proposition 2, this permutation belongs to S. Since that will be true for every term of π_n, all its proper subpermutations will belong to S.

Each pair (σ, p) of $\mathcal{P}$ is proved to have the correct property by a similar approach. We suppose that $\sigma' = s_1 s_2 s_3 s_4 s_5 s_6 \neq \sigma$ with $p \in W(\sigma')$ and derive a contradiction.

Case 1: $\sigma = 2\,9\,5\,1\,4\,3$ and $p = E_n$.

Here s_2 lies in one of $F_n G_n H_n I_n$, A_n, B_n or $C_n D_n$.

Suppose that s_2 is in $F_n G_n H_n I_n$. Consider the case where s_2 occurs before 4. We are then forced to have $s_2 = 9$ which in turn forces $s_1 s_3 s_4 s_5 s_6 = 2\,5\,1\,4\,3$ and so $\sigma' = \sigma$ giving a contradiction. Therefore s_2 has to be to the right of 4 in $F_n G_n H_n I_n$, that is in a subsequence of π_n that avoids 321. Therefore, regardless of whether s_3 is in $F_n G_n H_n I_n$ or in $C_n D_n$, we are forced to have s_4 in $C_n D_n$ which in turn forces s_5 to the right of E_n. This contradicts p having to be contained in $W(\sigma')$.

If s_2 is in $A_n C_n D_n$ then we are forced to have s_3 and s_4 in $C_n D_n$ and, since $C_n D_n$ is decreasing, s_5 to the right of E_n, a contradiction.

If we suppose that s_2 is in B_n then s_1 has to be outside B_n otherwise we have $s_1 = 4n + 10$ and $s_2 = 4n + 14$ which forces $\sigma' = B_n E_n$. Now, we have $s_1 < B_n$ and so s_4 must also be outside B_n, which forces s_4 into $C_n D_n$. Therefore, s_5 is to the right of D_n and so p cannot be in $W(\sigma')$.

Therefore, no such σ' exists and so $W(\sigma)$ is the only witnessed segment containing p.

Case 2: $\sigma = B_n E_n$ and $p = 9$.

The argument is similar to that in Case 1. We cannot have s_2 being one of $E_n, 2, 9$ for then p could not belong to $W(\sigma')$; nor can we have s_2 as one of $5, 1, 4$ since s_2 has value at least 6. Hence we must have s_2 in either $A_n B_n C_n D_n$ or in $F_n G_n H_n I_n$ and, in this case, to the right of symbol 4. If s_2 lies in $F_n G_n H_n I_n$ we can follow the argument given in Case 1 resulting in s_5 being to the right of E_n, which forces s_6 to the right of 9. It then follows that $W(\sigma')$ does not contain p, a contradiction.

Again following the argument in Case 1, if s_2 in $A_n C_n D_n$ then s_5 is to the right of E_n giving a contradiction as above. Finally, if s_2 belongs to B_n

then, by Case 1, s_1 must be outside B_n otherwise $\sigma' = B_nE_n = \sigma$; this results in s_5 being to the right of D_n, which in turn forces s_6 to the right of 2. Therefore, $W(\sigma')$ cannot contain 9

Case 3: $\sigma = 4i+3 \mid E_n \mid 4i+9 \mid 4i+2 \mid 4i+8 \mid 4i+7$ and $p = 4i+3$ for any $i \in [0, n-1]$.

First note that s_2 cannot be in C_nD_n since this would force s_1 to the left of C_n and s_5 to the right of D_n.

Next suppose that $s_2 = E_n$. We then have s_1 in D_n, that is, $s_1 = 4j+3$ where $j \le i$. We cannot have s_4 outside $F_nG_nH_nI_n$ as this forces s_4 to be between s_1 and E_n. Therefore s_4 is in $F_nG_nH_nI_n$ and as a result, so is s_3.

Now, s_3 has to be in $F_nG_nH_nI_n$ such that $s_3 \ge 4j+6$, that is s_3 is contained in ρ where

$$\rho = 4j+9 \mid J_{j+2} \mid J_{j+3} \mid \cdots \mid J_n \mid G_n \mid H_n \mid I_n.$$

Furthermore, s_4 has to be to the right of s_3 and less than $4j+3$, giving $s_4 = 4j+2$ as the only candidate; this in turn gives $s_3 = 4j+9$. Finally, $F_nG_nH_nI_n$ avoids 321 so we must have s_6 in $A_nB_nC_nD_n$ and left of s_1; this forces $s_6 = 4j+7$ and in turn, $s_5 = 4j+8$. We now have $W(\sigma') = 4j+3$ and so $j = i$. Therefore, $\sigma = \sigma'$ and we have a contradiction.

The argument in Case 1 now handles the remaining possibility of s_2 being in $F_nG_nH_nI_n$, A_n or B_n to show that no σ' exists.

Case 4: $\sigma = C_nE_nG_nI_nA_n$, $p = C_n$.

Here s_2 can be in any one of E_n, $F_nG_nH_nI_n$, A_n or B_n. As in case 3, s_2 cannot be in D_n.

Suppose $s_2 = E_n$, and that $s_1 = C_n$. We then require s_3 to be on the interval $(C_n+2, E_n) = [4n+6, 4n+10]$. Therefore, s_3 is contained in $4n+8 \mid I_n \mid A_n \mid 4n+10$ and so $s_3s_5s_6$ is a subsequence of

$$4n+8 \mid H_n \mid I_n \mid A_n \mid 4n+10;$$

this forces us to have $s_3s_4s_5s_6 = G_n \mid I_n \mid A_n$, which gives $\sigma' = \sigma$, a contradiction. On the other hand if $s_1 = 4j+3$ for some $j \in [0, n-1]$ then $s_6 = 4j+7$ by Case 3 and so p cannot be contained in $W(\sigma')$.

Again, the argument in Case 1 handles the remaining possibility of s_2 being in $F_nG_nH_nI_n$, A_n or B_n and we can therefore conclude that σ' does not exist.

Now we confirm that, for every term t of π_n, we can find some $(\sigma, p) \in \mathcal{P}$ such that $t \in \sigma$ with $t \neq p$ (the latter condition will be automatic so long as t is not the first term of σ). We view each term of π_n as of the form $4j+h$, where $j \in [1, n+3]$ if $h = 0$, $j \in [0, n+3]$ if $h \in \{1, 2\}$ and $j \in [0, n+2]$ if $h = 3$. Table 1 shows which cases handle which terms of π_n

Since all terms of π_n are accounted for the proof is complete. ∎

		Cases			
		1	**2**	**3**	**4**
h value	**0**	$j=1$	$j=n+3$	$j\in[2,n+1]$	$j=n+2$
	1	$j\in[0,2]$	$j\in[n{+}2,n{+}3]$	$j\in[1,n+1]$	N/A
	2	$j=0$	$j\in[n{+}2,n{+}3]$	$j\in[1,n-1]$	$j\in[n,n+1]$
	3	$j=0$	$j=n+2$	$j\in[1,n]$	$j=n+1$

Table 1: Table of which cases handle which terms of π_n.

3 Enumeration

When gathering enumeration results for cyclic closures it is helpful to note that 4 of the usual symmetries of the subpermutation relation (the ones generated by reversal, and complementation) respect cyclic closure. For the cyclic closures of Av(σ) where σ has length 3, this means that only $\sigma = 321$ and $\sigma = 231$ need be considered. We recall that, for each of these permutations, Av(σ) is enumerated by the Catalan sequence (c_n).

Theorem 9 *cc(Av(231)) is enumerated by*

$$n\,(c_n - c_{n-1} - c_{n-2} - \cdots - c_1)$$

for $n \geq 2$*.*

Proof: Every permutation of Av(231) of length n lies in a rotation class consisting of its n (different) cyclic rotations and every permutation of cc(Av(231)) arises in this way. Thus the permutations of degree n of cc(Av(231)) fall into disjoint rotation classes of size n. We shall find, in each such rotation class, a distinguished permutation of Av(231). Then we shall count the totality of such distinguished permutations and multiply the total by n.

Consider any one such rotation class and let σ be a permutation of it that lies in Av(231). Put $\sigma = \alpha n \beta$. Since $231 \not\preceq \sigma$ we have $\alpha < \beta$. Let $\beta = n-1, n-2, \ldots, n-k+1, \theta$ where either θ is empty or $\theta = u\phi$ with $u < n-k$. In the former case either α does not begin with $n-k$ or $\alpha = n-k\delta$ and the rotation $\delta n, n-1, \ldots, n-k$ also lies in Av(231). In the latter case (as $\alpha < u\phi$) the rotation $u\phi\alpha n, n-1, \ldots, n-k+1 \in$ Av(231).

These remarks show that, as the distinguished representative of any rotation class, we can take a permutation of the form

$$\sigma = \alpha n, n-1, \ldots, n-k+1$$

where, if α is non-empty, then α does not begin with its maximal symbol. The number of permutations of this type is easily seen to be

$$c_n - c_{n-1} - c_{n-2} - \cdots - c_1$$

since the only permutations we do not count are those of the form $\alpha n\beta$ where α begins with its maximum symbol and $\beta = n-1, n-2, \ldots, n-k+1$ (there being c_{n-k-1} of this type). ∎

Theorem 10 *$cc(Av(321))$ is enumerated by*

$$n\left(c_n - 2^n + \binom{n}{2} + 2\right)$$

for $n \geq 4$.

Proof: As in the previous proof we shall find a distinguished permutation of Av(321) in every rotation class of permutations of cc(Av(321)) and then count these distinguished permutations.

Put $Y = \text{Av}(321, 2143)$ and consider first the permutations of $\text{Av}(321) \setminus Y$. Such a permutation σ has a subsequence $s_2 s_1 s_4 s_3$ isomorphic to 2143 and we may write

$$\sigma = \alpha s_2 \beta s_1 \gamma s_4 \delta s_3 \epsilon$$

It is routine to check that there is no other cyclic rotation of σ that avoids 321. Hence permutations of $\text{Av}(321) \setminus Y$ are the only permutations of Av(321) in their rotation class.

Permutations of Y, however, are known [1] to have one of the forms shown in Figure 3 and, from this, it is easy to see that they have a rotation of the form 1γ that also lies in Y, and we shall take that to be the distinguished permutation of its rotation class.

Let (y_n) enumerate the class Y. We have $c_n - y_n$ distinguished permutations of length n in $\text{Av}(321) \setminus Y$ and y_{n-1} distinguished permutations of length n in Y. Thus cc(Av(321)) is enumerated by $n(c_n - y_n + y_{n-1})$. However, it is known [6] that

$$y_n = 2^{n+1} - 2n - 1 - \binom{n+1}{3}$$

and the result now follows.

∎

The methods employed in the proofs of the two previous theorems can be applied in other cases too. As examples of the results that can be obtained we give, in Table 2, a summary of enumerations of all inequivalent cyclic closures of classes of the form $\text{Av}(\alpha, \beta)$ where α and β each have length 3. The final line of the table treats Av(3142, 2413), the class of separable permutations, which was first introduced in [4]. Both this class and its enumeration by the Schröder sequence (s_n) have since made several appearances in the theory of subpermutations [2, 7].

Finally, we remark that enumerating the cyclic closures of $\text{Av}(\sigma)$ with $|\sigma| \geq 3$ is probably at least as hard as enumerating the original classes themselves. Therefore, in view of the results proved in [3], perhaps the next step would be the enumerate the cyclic closure of Av(1342).

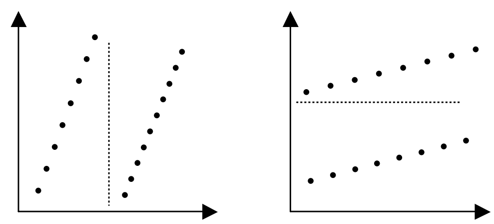

Figure 3: The two types of permutations in Av(321, 2143)

Class	Enumeration
cc(Av(123, 132))	$n(2^{n-1} - 2)$
cc(Av(123, 231))	$n(n^2 - 3n + 4)/2$
cc(Av(123, 321))	0 for $n \geq 5$
cc(Av(132, 213))	$n(2^{n-1} - n + 1)$
cc(Av(132, 231))	$n(2^{n-2})$
cc(Av(132, 312))	$n(2^{n-1} - n + 1)$
cc(Av(3142, 2413))	ns_{n-1}

Table 2: Some enumerations

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