# SOLOMON'S DESCENT ALGEBRA REVISITED

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ABSTRACT

Starting from a non-standard definition, the descent algebra of the symmetric group is investigated. Homomorphisms into the tensor product of smaller descent algebras are defined. They are used to construct the irreducible representations and to obtain the nilpotency index of the radical.

## 1. *Introduction*

In [7] Solomon introduced a remarkable family of algebras associated with Coxeter groups. For the special case of symmetric groups, these algebras have the following combinatorial description.

Let $\sigma$ be any permutation of the symmetric group $S_n$. The *signature* of $\sigma$ is the sequence of signs of the expressions $(i+1)^\sigma - i^\sigma$, $1 \leqslant i \leqslant n-1$. For example, the permutation [3 1 2 4] (in cycle notation (1 3 2)(4)) has signature $\varepsilon = [- + +]$. The $n!$ permutations of $S_n$ fall into $2^{n-1}$ disjoint signature classes. For each signature class $\varepsilon$ we let $A_\varepsilon$ denote the sum (in the group algebra of $S_n$ over some field $K$) of all elements in this signature class. Solomon's theorem states that, for any two signatures $\varepsilon, \eta$, $A_\varepsilon A_\eta$ is a linear combination (with non-negative integer coefficients) of signature class sums. Thus the signature class sums span a subalgebra of the group algebra of dimension $2^{n-1}$. Other proofs of this theorem were given in [3, **6**, **1**]. Following [**5**], we call this the *descent algebra* $\Sigma_n$.

Solomon also proved some results about the radical of $\Sigma_n$, but despite the intriguing definition of $\Sigma_n$ nothing further appeared until the detailed study by Garsia and Reutenauer [**5**]. They found some other natural bases for $\Sigma_n$ and used them to derive the indecomposable modules for $\Sigma_n$ and the Cartan invariants. Very recently [**2**], some homomorphisms were defined between the descent algebras.

In this paper we take a different approach, determining some properties of $\Sigma_n$ from an alternative definition. Our results can be understood without most of the ingenious theory developed in [**5**]. Indeed, the reader who is willing to accept our definition of $\Sigma_n$ will find the exposition independent of both [**5**] and [**7**]. Specifically, we shall define homomorphisms from $\Sigma_n$ into the algebra tensor product $\Sigma_a \otimes \Sigma_b \otimes \ldots \otimes \Sigma_k$, where $[a, b, \ldots, k]$ is any composition of $n$ (an ordered collection of positive integers whose sum is $n$). From these homomorphisms we can give explicitly a full set of irreducible representations of $\Sigma_n$. As a byproduct of this construction we identify the radical of $\Sigma_n$, giving a new proof of Theorem 3 of [**7**], and we prove a new result about the radical which generalises Theorem 5.7 of [**5**]. Special cases of our homomorphisms yield some of the homomorphisms defined in [**2**].

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For any composition $p$ of $n$ we let $p^*$ denote the associated partition of $n$. We write $p \approx q$ for any two compositions with $p^* = q^*$. This equivalence relation clearly has $p(n)$ equivalence classes, one for each partition of $n$.

The set of partitions and the set of compositions of $n$ each admit a *refinement* partial order. For partitions it is defined by $\pi_1 \leqslant \pi_2$ if the set of parts of the partition $\pi_2$ can be obtained from the set of parts of $\pi_1$ by repeatedly replacing a pair of parts by their sum. For two compositions $p, q$, we define $p \leqslant q$ if the components of $q$ can be obtained from the components of $p$ by repeatedly replacing *adjacent* components by their sum.

Let $p = [a_1, a_2, \ldots, a_r]$ and $q = [b_1, b_2, \ldots, b_s]$ be any two of the $2^{n-1}$ compositions of $n$. Let $S(p, q)$ denote the set of all $r \times s$ matrices $Z = (z_{ij})$ with non-negative integer entries such that

(i) $\sum_j z_{ij} = a_i$, for each $i = 1, 2, \ldots, r$, and

(ii) $\sum_i z_{ij} = b_j$, for each $j = 1, 2, \ldots, s$.

Our definition of $\Sigma_n$ is as follows. It is the vector space spanned by a basis of elements $B_p$, one basis element for every composition $p$ of $n$, with multiplication defined by

$$B_p B_q = \sum_{Z \in S(p,q)} B_{[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]}.$$

One small remark on this definition is necessary. Strictly,

$$[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]$$

may not be a composition of $n$ because some of the components may be zero. However, we can identify it with the composition obtained by omitting the zero components. Because of this, some basis elements $B_r$ can occur with multiplicity greater than 1 on the right-hand side of the expression for $B_p B_q$.

EXAMPLE. If $n = 5$, $p = [2, 3]$ and $q = [2, 1, 2]$, then $S(p, q)$ is the set of matrices

$$\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

and $B_p B_q = B_{[2,2,1]} + B_{[1,1,2,1]} + B_{[1,1,1,1,1]} + B_{[1,1,1,2]} + B_{[2,1,2]}$.

It is a routine calculation to verify that this definition of multiplication is associative and that $B_{[n]}$ is a multiplicative identity. It follows from results in **[4]** (see also Proposition 1.1 of **[5]**) that this is indeed the same algebra as Solomon's descent algebra but, as mentioned already, we shall develop our results directly from the definition above, to make our paper self-contained. We begin with an easy consequence of the definition of multiplication.

LEMMA 1.1. *If $B_c$ occurs in $B_p B_q$ with non-zero multiplicity, then $c \leqslant p$. Moreover, if $p \leqslant q$, then $B_p$ occurs in $B_p B_q$ with non-zero multiplicity.*

*Proof.* Let $p = [a_1, a_2, \ldots, a_r]$ and $q = [b_1, b_2, \ldots, b_s]$. If $B_c$ occurs in $B_p B_q$ with non-zero multiplicity, there exists a matrix $Z \in S(p, q)$ such that the removal of the zero components of $[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]$ gives the composition $c$. But $\sum_j z_{ij} = a_i$ and so $c \leqslant p$. If $p \leqslant q$ then $p$ may be decomposed into segments $(a_1, \ldots, a_u), (a_{u+1}, \ldots, a_v), \ldots$, which sum to $b_1, b_2, \ldots$ respectively. The matrix

$$\begin{pmatrix} a_1 & 0 & \ldots & 0 \\ a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_u & 0 & \ldots & 0 \\ 0 & a_{u+1} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & a_v & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \vdots \end{pmatrix}$$

is a member of $S(p,q)$ and so $B_p$ occurs as a summand of $B_p B_q$.

## 2. *Homomorphisms*

For any composition $p = [a_1, a_2, \ldots, a_r]$ of $n$ we define a linear map

$$\varepsilon_p : \Sigma_n \to \Sigma_{a_1} \otimes \Sigma_{a_2} \otimes \ldots \otimes \Sigma_{a_r}$$

by

$$\varepsilon_p(B_q) = \sum_{Z \in S(p,q)} B_{[z_{11}, z_{12}, \ldots, z_{1s}]} \otimes B_{[z_{21}, \ldots, z_{2s}]} \otimes \cdots \otimes B_{[z_{r1}, \ldots, z_{rs}]}$$

(and the usual understanding about omitting any zero $z_{ij}$ applies). From the definition of $S(p,q)$, the right-hand side does indeed lie in $\Sigma_{a_1} \otimes \Sigma_{a_2} \otimes \ldots \otimes \Sigma_{a_r}$.

EXAMPLE. $\varepsilon_{[2,3]}(B_{[2,1,2]}) = B_{[2]} \otimes B_{[2,1]} + B_{[1,1]} \otimes B_{[2,1]} + B_{[1,1]} \otimes B_{[1,1,1]} + B_{[1,1]} \otimes B_{[1,2]} + B_{[2]} \otimes B_{[1,2]}$.

THEOREM 2.1. *$\varepsilon_p$ is an algebra homomorphism.*

*Proof.* Let $p = [a_1, a_2, \ldots, a_r]$ and let $u = [b_1, b_2, \ldots, b_s]$, $v = [c_1, c_2, \ldots, c_t]$. Then $B_u B_v = \Sigma_Y B_{[y_{11}, y_{12}, \ldots, y_{1t}, y_{21}, \ldots, y_{2t}, \ldots, y_{s1}, \ldots, y_{st}]}$, where the summation is over all $y_{jk} \geqslant 0$ such that $\Sigma_k y_{jk} = b_j$, $\Sigma_j y_{jk} = c_k$. Thus

$$\varepsilon_p(B_u B_v) = \sum_Y \varepsilon_p(B_{[y_{11}, y_{12}, \ldots, y_{1t}, y_{21}, \ldots, y_{2t}, \ldots, y_{s1}, \ldots, y_{st}]}),$$

and this is equal to

$$\sum_{Y,Z} B_{[z_{111}, z_{112}, \ldots, z_{11t}, z_{121}, \ldots, z_{12t}, \ldots, z_{1st}]} \otimes B_{[z_{211}, \ldots, z_{2st}]} \otimes \cdots \otimes B_{[z_{r11}, \ldots, z_{rst}]},$$

where the summation is over all $y_{ij}$ and $z_{ijk}$ such that $\Sigma_{j,k} z_{ijk} = a_i$, $\Sigma_i z_{ijk} = y_{jk}$ and this is the same as a summation over all $z_{ijk}$ such that

$$\sum_{j,k} z_{ijk} = a_i, \quad \sum_{i,k} z_{ijk} = b_j, \quad \sum_{i,j} z_{ijk} = c_k.$$

On the other hand,

$$\varepsilon_p(B_u) = \sum_{X \in S(p,u)} B_{[x_{11}, x_{12}, \ldots, x_{1s}]} \otimes B_{[x_{21}, \ldots, x_{2s}]} \otimes \ldots \otimes B_{[x_{r1}, \ldots, x_{rs}]}$$

and

$$\varepsilon_p(B_v) = \sum_{Y \in S(p,v)} B_{[y_{11}, y_{12}, \ldots, y_{1t}]} \otimes B_{[y_{21}, \ldots, y_{2t}]} \otimes \ldots \otimes B_{[y_{r1}, \ldots, y_{rt}]},$$

and so

$$\varepsilon_p(B_u)\,\varepsilon_p(B_v) = \sum_{X,Y} B_{[x11, x_{12}, \ldots, x_{1s}]} B_{[y_{11}, y_{12}, \ldots, y_{1t}]} \otimes \ldots \otimes B_{[x_{r1}, \ldots, x_{rs}]} B_{[y_{r1}, \ldots, y_{rt}]}.$$

Now the $i$th tensor component in a typical summand of this expression has the form $B_{[x_{i1}, x_{i2}, \ldots, x_{is}]} B_{[y_{i1}, y_{i2}, \ldots, y_{it}]} = \Sigma B_{[z_{i11}, z_{i12}, \ldots, z_{i1t}, z_{i21}, \ldots, z_{i2t}, \ldots, z_{ist}]}$, the summation being over all $z_{ijk}$ with $\Sigma_k z_{ijk} = x_{ij}$ and $\Sigma_j z_{ijk} = y_{ik}$. It now follows easily that $\varepsilon_p(B_u)\,\varepsilon_p(B_v) = \varepsilon_p(B_u B_v)$, and so $\varepsilon_p$ is an algebra homomorphism.

LEMMA 2.2. *The dimension of the right ideal $B_p \Sigma_n$ and the dimension of the image of $\varepsilon_p$ are equal.*

*Proof.* Let $p = [a_1, a_2, \ldots, a_r]$. By one of the fundamental properties of the tensor product there is a linear mapping $\zeta$ from $\Sigma_{a_1} \otimes \Sigma_{a_2} \otimes \ldots \otimes \Sigma_{a_r}$ to $\Sigma_n$ which maps each basis element $B_{[z_1, z_2, \ldots, z_s]} \otimes B_{[z_{s+1}, \ldots, z_t]} \otimes \ldots \otimes B_{[z_u, \ldots, z_v]}$ to $B_{[z_1, z_2, \ldots, z_s, z_{s+1}, \ldots, z_v]}$, and the mapping $\zeta$ is evidently one-to-one (it is, however, not an algebra homomorphism). Since $\zeta(\varepsilon_p(B_q)) = B_p B_q$ we have $\zeta(\varepsilon_p(\Sigma_n)) = B_p \Sigma_n$. The lemma follows since $\zeta$ is one-to-one.

In fact, rather more than this lemma is true. Let $K_p$ denote the kernel of $\varepsilon_p$ (a two-sided ideal). Then we have the following.

THEOREM 2.3. $\Sigma_n = K_p \oplus B_p \Sigma_n$.

*Proof.* By the previous lemma it suffices to prove that $K_p \cap B_p \Sigma_n = 0$, equivalently that $\varepsilon_p$ maps $B_p \Sigma_n$ monomorphically. The argument of the previous lemma shows that $\dim(\varepsilon_p(B_p \Sigma_n)) = \dim(B_p^2 \Sigma_n)$, and so it is enough to prove that $B_p^2 \Sigma_n = B_p \Sigma_n$. To do this we consider the action of right multiplication by $B_p$ on the subspace $T_p = \langle B_q \mid q \leqslant p \rangle$ which, by Lemma 1.1, is a right ideal. If we list the compositions of $n$ in a linear order that extends the refinement order, Lemma 1.1 shows that the action of $B_p$ has a triangular matrix with non-zero diagonal entries and so is a non-singular action; therefore the linear transformation on $T_p$ induced by $B_p$ satisfies an equation $f(\lambda) = 0$ with non-zero constant coefficient. It follows that $B_p f(B_p) = 0$ in $\Sigma_n$ and so $B_p$ is in the right ideal generated by $B_p^2$, from which the result follows.

Suppose that $V_1, \ldots, V_r$ is a family of algebras, that $p = [a_1, a_2, \ldots, a_r]$ is a composition of $n$, and that there are algebra homomorphisms $\phi_i : \Sigma_{a_i} \to V_i$. Then the composite mapping $\varepsilon_p$ followed by $\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_r$ is an algebra homomorphism from $\Sigma_n$ into $V_1 \otimes V_2 \otimes \ldots \otimes V_r$. We shall use this fact in two ways: to give a simple description of some of the homomorphisms defined in [2] and, in the next section, to derive a full set of irreducible representations for $\Sigma_n$.

Let $T_n$ be the subspace of $\Sigma_n$ spanned by all $B_p$ with $p \neq [n]$. By Lemma 1.1, $T_n$ is closed under multiplication, and since it is invariant under left and right multiplication by $B_{[n]}$, the identity element of $\Sigma_n$, it is also a two-sided ideal. Hence there is a homomorphism $\phi_{[n]}: \Sigma_n \to K$ which maps $B_{[n]}$ to 1 and all other $B_p$ to 0. For any composition $[a, b]$ of $n$, consider the composite mapping $\varepsilon_{[a,b]}$ followed by $1_{\Sigma_a} \otimes \phi_{[b]}$ which maps $\Sigma_n$ to $\Sigma_a \otimes K \cong \Sigma_a$. The image under this map of $B_q$, where $q = [c_1, \ldots, c_s]$, is $\Sigma B_{[a_1, a_2, \ldots, a_s]} \otimes \phi_{[b]}(B_{[b_1, b_2, \ldots, b_s]})$, where the summation is over all $a_1, \ldots, a_s, b_1, \ldots, b_s$ with $\Sigma a_i = a$, $\Sigma b_i = b$ and $a_i + b_i = c_i$ for each $i$. By definition of $\phi_{[b]}$ this is equal to $\Sigma B_{[a_1, a_2, \ldots, a_s]}$ summed over all $a_1, \ldots, a_s$ in which $a_i = c_i$ for all but one value of $i$ and $a_i = c_i - b$ for the remaining value of $i$. This is precisely the definition of the homomorphism denoted by $\Delta_b$ in [2].

## 3. *Irreducible representations*

If $p = [a_1, a_2, \ldots, a_r]$ is a composition of $n$, let $\delta_p$ be the composite mapping $\varepsilon_p$ followed by $\phi_{[a_1]} \otimes \phi_{[a_2]} \otimes \ldots \otimes \phi_{[a_r]}$, a homomorphism from $\Sigma_n$ into $K \otimes K \otimes \ldots \otimes K \cong K$.

LEMMA 3.1. (i) *$\delta_p(B_q) = 0$ unless $p^* \leqslant q^*$.*

(ii) *If $p \approx q$ then $\delta_p(B_q) = t_1!\, t_2! \ldots t_n!$, where $t_i$ is the number of components of $p$ which are equal to $i$.*

*Proof.* Let $p = [a_1, a_2, \ldots, a_r]$. A term on the right-hand side of

$$\delta_p(B_q) = \sum_{Z \in S(p,q)} \phi_{[a_1]}(B_{[z_{11}, z_{12}, \ldots, z_{1s}]}) \otimes \phi_{[a_2]}(B_{[z_{21}, \ldots, z_{2s}]}) \otimes \cdots \otimes \phi_{[a_r]}(B_{[z_{r1}, \ldots, z_{rs}]})$$

can be non-zero only if, for each $1 \leqslant i \leqslant r$, the only non-zero integer among $z_{i1}, z_{i2}, \ldots, z_{is}$ is $a_i$. In addition, each component of the composition $q$ is a column sum of the matrix $Z$ and so is a sum of components of $p$. This proves (i). For the second part, observe that if $p \approx q$ then each such matrix $Z$ will have exactly one non-zero entry in each row and column. The total number of such matrices is $t_1!\, t_2! \ldots t_n!$, and each contributes 1 to $\delta_p(B_q)$.

LEMMA 3.2. *If $p \approx q$ then $\delta_p = \delta_q$ and the set of $p(n)$ $\delta_p$ (one for each equivalence class) is a linearly independent set.*

*Proof.* (i) If $p \approx q$ then, for any composition $c$, the matrices of $S(p, c)$ are related to those of $S(q, c)$ by applying a fixed permutation of their rows. Thus the summands on the right-hand side of

$$\varepsilon_p(B_c) = \sum_{Z \in S(p,c)} B_{[z_{11}, z_{12}, \ldots, z_{1s}]} \otimes B_{[z_{21}, \ldots, z_{2s}]} \otimes \ldots \otimes B_{[z_{r1}, \ldots, z_{rs}]}$$

are related to the summands on the right-hand side of the corresponding expression for $\varepsilon_q(B_c)$ in that the tensor components are permuted by a fixed permutation. But $K$ is commutative, so

$$\delta_p(B_c) = \sum_{Z \in S(p,c)} \phi_{a_1}(B_{[z_{11}, z_{12}, \ldots, z_{1s}]}) \otimes \phi_{a_2}(B_{[z_{21}, \ldots, z_{2s}]}) \otimes \ldots \otimes \phi_{a_r}(B_{[z_{r1}, \ldots, z_{rs}]}) = \delta_q(B_c).$$

To prove (ii), let $\Sigma_{q \in Q}\, a_q \delta_q = 0$ be a linear dependence relation where $Q$ is a set of inequivalent compositions with all $a_q \neq 0$. If the dependence relation is non-trivial, we can select some $r \in Q$ with $r^*$ minimal among the set of partitions $\{q^* \mid q \in Q\}$. Then, for all $q \in Q$, $q \neq r$, we have $\delta_q(B_r) = 0$ (otherwise $q^* \leqslant r^*$ by Lemma 3.1 and $r^*$ would not be minimal), and therefore $a_r \delta_r(B_r) = 0$, contradicting Lemma 3.1.

LEMMA 3.3. $\langle B_p - B_q \mid p \approx q \rangle = \bigcap \ker \delta_c$.

*Proof.* The two sides of the equation each have codimension $p(n)$ in $\Sigma_n$, and so it is sufficient to prove that the left-hand side is contained in the right-hand side, and for this it is enough to prove that $\delta_c(B_p) = \delta_c(B_q)$ for any $c$, and any $p \approx q$. If $c = [a_1, a_2, \ldots, a_r]$ and $p$ has $s$ components, then

$$\delta_c(B_p) = \sum_{Z \in S(c,\, p)} \phi_{a_1}(B_{[z_{11}, z_{12}, \ldots, z_{1s}]}) \otimes \phi_{a_2}(B_{[z_{21}, \ldots, z_{2s}]}) \otimes \ldots \otimes \phi_{a_r}(B_{[z_{r1}, \ldots, z_{rs}]}).$$

In this sum, the only matrices $Z$ which give a non-zero contribution are those with a single non-zero entry $a_i$ in the $i$th row (and column sums giving the components of $p$), and for each such matrix the contribution is 1. But the matrices in $S(c, q)$ of this form are obtained from those in $S(c,p)$ by applying a fixed column permutation. Therefore $\delta_c(B_p) = \delta_c(B_q)$, as required.

THEOREM 3.4. (i) $\langle B_p - B_q \mid p \approx q \rangle = R_n$, *the radical of* $\Sigma_n$, *and* (ii) $R_n^{n-1} = 0$.

*Proof.* Temporarily denote the left-hand side of (i) by $L_n$. By the previous lemma, $\Sigma_n / L_n$ is semi-simple so $R_n \subseteq L_n$. We shall prove the reverse inclusion by proving that $L_n$ is nilpotent of index at most $n-1$, and this will prove both parts of the theorem.

Let $p = [a_1, a_2, \ldots, a_r]$, $q = [b_1, b_2, \ldots, b_s]$. By Lemma 1.1, the only terms $B_c$ that can occur in the product $B_p B_q$ are terms $B_p$ and terms $B_c$ where $c$ has more components than $p$. Consider the coefficient of $B_p$ in $B_p B_q$. By definition it is equal to the number of matrices $Z \in S(p, q)$ in which $[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]$ reduces to $[a_1, a_2, \ldots, a_r]$ when zero components are deleted. Such matrices are precisely those with a single non-zero entry in the $i$th row equal to $a_i$ and whose column sums give the components of $q$. There is an obvious one-to-one correspondence between this subset of $S(p, q)$ and the analogous subset of $B_p B_t$ if $t \approx q$ (given by permuting the columns). It follows that, if $t \approx q$, $B_p(B_q - B_t)$ is a linear combination of elements $B_c$ where each composition $c$ has strictly more components than $p$. But now an obvious induction shows that if $x_1, \ldots, x_k$ are all of the form $B_q - B_t$, $t \approx q$, that is they are members of the spanning set of $L_n$, then $B_p x_1 \ldots x_k$ is a linear combination of elements $B_c$ where each composition $c$ has at least $r + k$ components. In particular, since $B_{[n]}$ is the identity element and $[n]$ has one component, $x_1 x_2 \ldots x_n = 0$. This already proves that $L_n^n = 0$. However, we also know that $L_n^{n-1}$ is contained in the 1-dimensional space generated by $B_{[1,1,\ldots,1]}$, as $[1, 1, \ldots, 1]$ is the only composition with one part; but $B_{[1,1,\ldots,1]}$ is not nilpotent, and so $L_n^{n-1} = 0$, as required.

REMARKS. 1. It was proved in [5] that every element $x$ of $R_n$ satisfies $x^{n-1} = 0$.

2. Part (i) of the theorem was proved in [7] for general Coxeter groups; by somewhat similar methods, part (ii) may also be proved in this more general setting.

COROLLARY 3.5. *The nilpotency index of* $R_n$ *is* $n-1$.

*Proof.* By the previous theorem, it suffices to show that $R_n^{n-2} \neq 0$. Let $w = B_{[1,n-1]} - B_{[n-1,1]}$ and let $D(a,b) = B_{[1^a,t,1^b]}$, where $a+b+t = n$. Clearly, $\{D(a,b) \mid a+b < n-1\}$ is a linearly independent set. From the rule for multiplication it follows directly that $D(a,b)\,w = D(a+1,b) - D(a,b+1)$. Then an easy induction shows that

$$w^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} D(r-k,k).$$

Thus $w^{n-2} \neq 0$, and the proof is complete.

Since $\Sigma_n/R_n$ is of dimension $p(n)$ and we have identified $p(n)$ linear (therefore irreducible) representations of $\Sigma_n$, we have the following result.

THEOREM 3.6. *The $p(n)$ distinct linear representations $\delta_p$ are a full set of irreducible representations for $\Sigma_n$.*

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