The p-modular Descent Algebra of the Symmetric Group

S.J. van Willigenburg[*]
M.D. Atkinson
School of Mathematical and Computational Sciences
North Haugh, St Andrews KY16 9SS, UK

October 1, 1996

Abstract

The descent algebra of the symmetric group, over a field of non-zero characteristic p, is studied. A homomorphism into the algebra of generalised p-modular characters of the symmetric group is defined. This is then used to determine the radical, and its nilpotency index. It also allows the irreducible representations of the descent algebra to be described.

1 Introduction

In 1976, Louis Solomon defined a family of algebras associated with Coxeter groups [6]. In the case of symmetric groups their definition can be expressed as follows:

If σ is any permutation in the symmetric group S_n written in image form (e.g. [1342]) then the *signature* of σ is the sequence of signs $\{x_i\}_{i=1}^{n-1}$ where

$$ x_i = \begin{cases} + & \text{if } (i+1)^\sigma - i^\sigma < 0 \\ - & \text{if } (i+1)^\sigma - i^\sigma > 0 \end{cases} $$

For example, [1342] has the signature $\varepsilon = [++-]$. Such signatures partition the $n!$ permutations of S_n into 2^{n-1} disjoint signature classes, and we denote the sum of all elements in a given signature class, ε, by A_ε. Solomon proved that, for any two signatures ε, η, $A_\varepsilon A_\eta$ is a linear combination (with non-negative integer coefficients) of signature class sums. Hence the signature class sums span a sub-algebra of the group algebra of dimension 2^{n-1} which has become known as the *descent algebra* Σ_n [3].

The algebra Σ_n is not semi-simple. Indeed, Solomon proved that the dimension of its radical is $2^{n-1} - p(n)$ (where $p(n)$ is the partition function). Garsia

---

and Reutenauer, in their extensive paper [3], gave another proof of this result; they also derived other natural bases for Σ_n and determined the Cartan invariants. In other work on Σ_n, Atkinson [1] defined a family of homomorphisms on Σ_n, including an epimorphism from Σ_n to Σ_{n-1}, and proved that the nilpotency index of the radical is $n-1$; and very recently Gelfand *et al* [4] have used the descent algebra in a key way in their work on non-symmetric functions. In all these papers, Σ_n has been studied as an algebra over a field of characteristic zero. However, since the structure constants of the algebra are integers, it is also possible to define the descent algebra over fields $\mathcal{F}_p$ of any prime order p. For values of $p > n$ all the above results extend virtually unchanged but, as we shall see in this paper, $p \leq n$ gives rise to a more complicated situation. In this case the dimension of the radical depends on p as well as n. Nevertheless we are able to identify the radical (by giving a natural basis for it), determine its nilpotency index, and describe the irreducible representations of the descent algebra.

It is convenient to work with the alternative definition of Σ_n given below (and justified in [3]) in which Σ_n is defined by a basis $\{B_q\}$ indexed by compositions of n.

If $q = [a_1, a_2, \ldots, a_s]$ and $r = [b_1, b_2, \ldots, b_t]$ are compositions of n we define $S(q, r)$ to be the set of all $s \times t$ matrices $Z = (z_{ij})$ with non-negative integer entries such that

1. $\sum_j z_{ij} = a_i$ for each $i = 1, 2, \ldots, s$

2. $\sum_i z_{ij} = b_j$ for each $j = 1, 2, \ldots, t$

Multiplication in Σ_n is then defined by the rule

$$B_q B_r = \sum_{Z \in S(q,r)} B_{[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]} \tag{1}$$

Remark Due to some z_{ij} possibly being zero,

$$[z_{11}, z_{12}, \ldots, z_{1s}, z_{21}, \ldots, z_{2s}, \ldots, z_{r1}, \ldots, z_{rs}]$$

may not be a composition, but it can be identified with the composition obtained by omitting zero components and, because of this, the multiplicity of a basis element B_s in the right hand side of Equation 1 may be greater than one.

Example 1 If $n = 4, q = [2, 2], r = [2, 1, 1]$ then $S(q, r)$ is the set of matrices

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Hence,

$$B_q B_r = B_{[2,1,1]} + B_{[1,1,2]} + 2B_{[1,1,1,1]}$$

2

In order to study the characteristic p analogue of Σ_n, we define $\mathscr{Z}_n$ to be the subring of Σ_n consisting of all integral combinations of the basis elements $\{B_q\}$, and consider its ideal $\mathcal{P}_n = p\mathscr{Z}_n$. We define $\Sigma(n, p)$ to be the quotient ring $\mathscr{Z}_n/\mathcal{P}_n$; $\Sigma(n, p)$ is clearly an algebra over $\mathcal{F}_p$ which we term the p-modular descent algebra. Of course, $\Sigma(n, p)$ is the algebra that would arise if the field of coefficients in the definition of Σ_n had been taken as $\mathcal{F}_p$.

We let $\rho_1 : \mathscr{Z}_n \to \Sigma(n, p)$ be the natural homomorphism with kernel $\mathcal{P}_n$ and write $\overline{B}_q = \rho_1(B_q)$. The set $\{\overline{B}_q\}$ is obviously a basis for $\Sigma(n, p)$ and, as already implied, the multiplication rule for $\overline{B}_q\overline{B}_r$ is the same as for $B_q B_r$ except that coefficients are reduced modulo p. Thus, as a consequence of Example 1, in $\Sigma(4, 2)$, $\overline{B}_{[2,2]}\overline{B}_{[2,1,1]} = \overline{B}_{[2,1,1]} + \overline{B}_{[1,1,2]}$.

Let $q = [a_1, a_2, \ldots, a_r]$ be a composition of n, let $H_q = S_{a_1} \times S_{a_2} \times \ldots \times S_{a_r}$ be the corresponding Young subgroup of S_n, let 1_q be the principal character of H_q, and let $\chi_q = 1_q^{S_n}$ be the Young character corresponding to q. Then the Z−module G_n consisting of all integral combinations of $\{\chi_q\}$ is, by the Mackey formula, closed under pointwise product and so has a ring structure. Solomon [6] proved that the linear map $\theta : \mathscr{Z}_n \to G_n$ defined by $\theta(B_q) = \chi_q$, for all compositions q, is a homomorphism of rings. This map was a key tool in Solomon's paper; he proved that its kernel $\mathcal{R}_n$ is spanned by all differences $B_q - B_r$, where q and r induce the same partition of n, and that $\mathcal{R}_n$ is nilpotent.

To extend these results to $\Sigma(n, p)$ we let ρ_2 be the map defined on generalised characters in G_n (all of which have integral values) which simply reduces the character values modulo p, and we let $G(n, p)$ denote the image of G_n under this map; clearly, $G(n, p)$ is a commutative algebra over $\mathcal{F}_p$. The kernel of the composite map

$$\mathscr{Z}_n \to G_n \to G(n, p)$$

obviously contains $\mathcal{P}_n$ and so induces an epimomorphism of $\mathcal{F}_p-$ algebras $\phi : \Sigma(n, p) \to G(n, p)$ which satisfies

$$\phi(\rho_1(x)) = \rho_2(\theta(x)) \text{ for all } x \in \Sigma_n$$

Writing $\tilde{\chi}_q$ for $\rho_2(\chi_q)$ we obtain, in particular, $\phi(\overline{B}_q) = \tilde{\chi}_q$. The homomorphism ϕ will enable us to describe $\mathcal{R}(n, p)$, the radical of $\Sigma(n, p)$, in a manner similar to the description in [6] of the radical of Σ_n.

We conclude this section by defining two binary relations on the set of compositions which we then use to describe some useful properties of the multiplication rule for $B_q B_r$.

If q and r are compositions of n which differ only in the order of their components then we write $q \approx r$. The relation $q \approx r$ is an equivalence relation on the compositions of n with, clearly, $p(n)$ equivalence classes.

There is also a partial order relation on the set of compositions. We write $q \preceq r$ if the components of q can be obtained from the components of r by repeatedly replacing adjacent components by their sum.

Definition 1 *Two matrices are said to be* column equivalent *if one can be obtained from the other by permuting the columns.*

Lemma 1 *Let B_q and B_r be basis elements of Σ_n and suppose that, in the composition r, the number of components equal to i is denoted by t_i. Then*

1. *If the coefficient of B_s in $B_q B_r$ is non-zero then $s \preceq q$*

2. *The coefficient of B_q in the product $B_q B_r$ is a multiple of $t_1! t_2! \ldots t_n!$ and this coefficient depends on the equivalence class of r only*

3. *If $q \approx r$, the coefficient of B_q in $B_q B_r$ is exactly $t_1! t_2! \ldots t_n!$*

PROOF The first statement follows from Lemma 1.1 of [1]. To prove the remaining statements let $q = [a_1, \ldots, a_u]$ and $r = [b_1, \ldots, b_v]$. A matrix $Z \in S(q, r)$ which contributes to the coefficient of B_q in $B_q B_r$ satisfies

$$\sum_j z_{ij} = a_i \text{ and } \sum_i z_{ij} = b_j$$

and the non-zero entries of the rows of Z, if read in serial order, yield $a_1, \ldots, a_u$. It follows that the ith row of Z has a single non-zero entry which is equal to a_i. Note also that, since all $b_j > 0$, every column of Z has at least one non-zero entry.

The set of matrices $\mathcal{Q}$ (if any) which satisfy these conditions falls into a number of column equivalence classes. Each of these classes has precisely $t_1! t_2! \ldots t_n!$ members since the set of columns of one of the matrices in $\mathcal{Q}$ with a common sum may be permuted arbitrarily. Thus the coefficient of B_q in $B_q B_r$ is indeed a multiple of $t_1! t_2! \ldots t_n!$. If s is some composition equivalent to r the set of matrices that is analogous to $\mathcal{Q}$ is related to $\mathcal{Q}$ by permuting columns. This proves the second statement. For the third statement we note that, when $q \approx r$, $\mathcal{Q}$ consists of exactly one column equivalence class since then the matrices will have exactly one non-zero entry in each column as well as each row. $\square$

Note that the conclusions of Lemma 1 hold also for basis elements $\overline{B}_q, \overline{B}_r$ of $\Sigma(n, p)$ except that the coefficients in question must be reduced modulo p.

2 The Form of the Radical and the Irreducible Representations of $\Sigma(n, p)$

Lemma 2 *$G(n, p)$ has dimension $g(n, p)$ over $\mathcal{F}_p$ where $g(n, p)$ is the number of conjugacy classes of p-regular elements in S_n.*

PROOF For each composition q and partition π let $m_{q\pi}$ be the value of the character $\tilde{\chi}_q$ on the conjugacy class of elements of S_n of cycle type π and let M be the $2^{n-1} \times p(n)$ matrix $[m_{q\pi}]$. Then $\dim G(n, p) = \text{rank } M$.

If π_1, π_2 are the partition cycle types of two elements of S_n with the same p-regular part then by §82 of [2] the columns of M which correspond to π_1, π_2 are equal. Thus rank $M \leq g(n, p)$.

To prove that rank $M \geq g(n, p)$ we list the rows of M so that the first $p(n)$ rows are indexed by a complete set of inequivalent compositions. We can then consider the $p(n) \times p(n)$ submatrix N consisting of these rows and index them by partitions. If the partitions indexing the rows and columns of N are listed lexico-graphically then N is a lower triangular matrix; furthermore, if $\pi = 1^{t_1} 2^{t_2} \ldots n^{t_n}$ is a typical partition then the (π, π) diagonal entry of N is $t_1! t_2! \ldots t_n! \mod p$ (which follows from the tabloid method of evaluating permutation characters [5], p41). By [5] p41 again there are $g(n, p)$ non-zero diagonal entries and so rank $M \geq g(n, p)$. □

Lemma 3 $\mathcal{R}(n, p)) \subseteq \ker \phi,$

PROOF The image of ϕ is a space of functions defined over a field and is therefore semi-simple. Consequently the two-sided nilpotent ideal $\phi(\mathcal{R}(n, p))$ must be zero. □

Theorem 1 $\Sigma(n, p)/\mathcal{R}(n, p)$ *is commutative.*

PROOF Since $\mathcal{R}_n$ is a nilpotent ideal of $\mathcal{Z}_n$, $\rho_1(\mathcal{R}_n)$ is a nilpotent ideal of $\Sigma(n, p)$, and therefore $\rho_1(\mathcal{R}_n) \subseteq \mathcal{R}(n, p)$. Hence there exists an ideal $\mathcal{S}_n$ of Σ_n, the pre-image of $\mathcal{R}(n, p)$, such that $\mathcal{R}_n \subseteq \mathcal{S}_n$ and $\mathcal{S}_n/\mathcal{P}_n \cong \mathcal{R}(n, p)$. Since $\Sigma(n, p) \cong \mathcal{Z}_n/\mathcal{P}_n$, $\Sigma(n, p)/\mathcal{R}(n, p) \cong \mathcal{Z}_n/\mathcal{S}_n$ is a homomorphic image of $\mathcal{Z}_n/\mathcal{R}_n \cong G_n$. Since the latter ring is commutative the theorem follows. □

Lemma 4 *Let $\overline{B}_r$ be a basis element of $\Sigma(n, p)$. Then $\overline{B}_r$ is nilpotent if and only if r has a component of multiplicity p or more.*

PROOF Suppose that r has t_i components equal to i. Set

$$I = \langle \overline{B}_q | q \preceq r \rangle$$

By Lemma 1 I is a right ideal of $\Sigma(n, p)$ and so right multiplication by $\overline{B}_r$ induces a linear transformation on I. We consider the matrix of this transformation with respect to the given basis $\overline{B}_{q_1} \ldots \overline{B}_{q_w}$ of I ordered so that $q_i \preceq q_j$ implies $i \leq j$. This matrix is, by Lemma 1, lower triangular with diagonal elements all equal to $t_1! t_2! \ldots t_n! \mod p$. Therefore the matrix is nilpotent if and only if one of the multiplicities t_i is p or more. If the matrix is not nilpotent then certainly $\overline{B}_r$ is not nilpotent. On the other hand, if the matrix is nilpotent then $I\overline{B}_r^t = 0$ for some t and so, as $\overline{B}_r \in I$, $\overline{B}_q^{t+1} = 0$. □

Lemma 5 $\dim \mathcal{R}(n, p)) \geq \dim \ker \phi.$

PROOF $\Sigma(n, p)/\mathcal{R}(n, p)$ is a commutative semi-simple algebra and so contains no non-zero nilpotent elements. Hence all nilpotent elements of $\Sigma(n, p)$ are contained in $\mathcal{R}(n, p)$.

The elements $B_q - B_r$ with $q \approx r$ of Σ_n lie in the radical of Σ_n ([6], Theorem 3) and so are all nilpotent. Hence their images $\overline{B}_q - \overline{B}_r$ are also nilpotent; they span a subspace U of $\mathcal{R}(n, p)$ of dimension $2^{n-1} - p(n)$.

If q is a composition with a component of multiplicity p or more then every composition r with $q \approx r$ also has this property. We choose a complete set A of inequivalent compositions with this property; clearly the members of A can be put in 1-1 correspondence with the set of *partitions* of n which have a part of multiplicity p or more. However, it is known that the number of such partitions is the same as the number of partitions which have a part divisible by p [5], p.41, and this number is $p(n) - g(n, p)$.

Finally we note that $\{\overline{B}_q | q \in A\}$, a set of nilpotent elements, is contained in $\mathcal{R}(n, p)$ and is linearly independent of the subspace U. Therefore

$$
\begin{aligned}
\dim \mathcal{R}(n, p) \quad &\geq \quad 2^{n-1} - p(n) + p(n) - g(n, p) \\
&= \quad 2^{n-1} - g(n, p) \\
&= \quad \dim \Sigma(n, p) - \dim G(n, p) \\
&= \quad \dim \ker \phi
\end{aligned}
$$

$\square$

We can now describe $\mathcal{R}(n, p)$ exactly.

Theorem 2 $\mathcal{R}(n, p) = \ker \phi$ *and is spanned by all* $\overline{B}_q - \overline{B}_r$ *with* $q \approx r$ *together with all* $\overline{B}_q$ *where* q *has a component of multiplicity* p *or more.*

PROOF Lemma 3 and Lemma 5 prove that $\mathcal{R}(n, p) = \ker \phi$. The proof of Lemma 5 then shows that $\mathcal{R}(n, p)$ not only contains but is actually spanned by all $\overline{B}_q - \overline{B}_r$ with $q \approx r$ together with all $\overline{B}_q$ where q has a component of multiplicity p or more. $\square$

From Theorem 2 it follows that $\dim \Sigma(n, p)/\mathcal{R}(n, p) = g(n, p)$ and so, by Theorem 1, $\Sigma(n, p)$ has $g(n, p)$ irreducible representations all of which are 1-dimensional. We may describe them as follows.

Let π be any partition of n and x any element of $\Sigma(n, p)$. Then $\phi(x)$ is a p-modular character of S_n and we let $\phi(x)^\pi$ be the value of this character on the conjugacy class corresponding to π. Define $\lambda_\pi : \Sigma(n, p) \to \mathcal{F}_p$ by

$$
\lambda_\pi(x) = \phi(x)^\pi \text{ for all } x \in \Sigma(n, p)
$$

It follows, since ϕ is a homomorphism and characters of S_n are added and multiplied pointwise, that λ_π is a (1-dimensional) representation of $\Sigma(n, p)$.

λ_π is determined by its values $\phi(\overline{B}_q^\pi) = \tilde{\chi}^\pi$ on the basis of $\Sigma(n, p)$ and, by ordering the basis, we can define a column vector D^π of these values. By the proof of Lemma 2 the matrix whose columns are the vectors D^π has rank $g(n, p)$. That lemma also shows that both the set of p-regular partitions and the set of partitions with no part of multiplicity p or more provides a suitable set of distinct columns that may be taken to define $g(n, p)$ distinct irreducible representations of $\Sigma(n, p)$.

3 The Nilpotency Index of the Radical

Let Y_m be the subspace of $\Sigma(n, p)$ spanned by all $\overline{B}_q$ where q has m or more components (for simplicity of notation we omit the reference to the dependency on n and p). Then

$$\Sigma(n, p) = Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_n \supseteq Y_{n+1} = 0$$

Lemma 6 $Y_m \mathcal{R}(n, p) \subseteq Y_{m+1}$

PROOF Let s be a composition with at least m components (so that $\overline{B}_s \in Y_m$) and consider the product $\overline{B}_s X$ for each of the spanning elements of $\mathcal{R}(n, p)$ given in Theorem 2. Such a product is, by Lemma 1, a linear combination of terms $\overline{B}_t$ with $t \preceq s$ but, as we now prove, the term $\overline{B}_s$ itself occurs with coefficient zero. There are two cases to consider:

1. $X = \overline{B}_q - \overline{B}_r, q \approx r$. By Lemma 1, the coefficients of $\overline{B}_s$ in both $\overline{B}_s \overline{B}_q$ and $\overline{B}_s \overline{B}_r$ are equal; thus, in $\overline{B}_s(\overline{B}_q - \overline{B}_r)$, the coefficient of $\overline{B}_s$ is zero.

2. $X = \overline{B}_r$ where r has t_i components equal to i with at least one t_i being p or more. Again, by Lemma 1 since $t_1! \ldots t_n!$ is zero in $\mathcal{F}_p$, the coefficient of $\overline{B}_s$ in $\overline{B}_s \overline{B}_r$ is zero.

It now follows that $Y_m X \subseteq Y_{m+1}$ for all $X \in \mathcal{R}(n, p)$ and this completes the proof. $\qquad\square$

Let $\mathcal{T}$ denote the subspace of $\mathcal{R}(n, p)$ generated by all $\overline{B}_q - \overline{B}_r$ with $q \approx r$ (again we omit the reference to the dependency on n and p). Since $\mathcal{T}$ is the image of R_n under the homomorphism ρ_1, $\mathcal{T}$ is a nilpotent ideal and therefore is contained in $\mathcal{R}(n, p)$.

Lemma 7 *1. If n is odd or $p \neq 2$ then $\mathcal{R}(n, p) \subseteq Y_2 \cap \mathcal{T} + Y_3$*

2. If n is even and $p = 2$ then $\mathcal{R}(n, p) \subseteq \langle \overline{B}_{[n/2, n/2]} \rangle + Y_2 \cap \mathcal{T} + Y_3$

PROOF Consider the spanning set for $\mathcal{R}(n, p)$ given in Theorem 2. An element $\overline{B}_q - \overline{B}_r$ with $q \approx r$ is non-zero only if q and r have at least 2 components and so such an element belongs to $Y_2 \cap \mathcal{T}$.

Consider an element $\overline{B}_q$ where the composition q has a component which occurs p times or more. If n is odd or $p \neq 2$ then q will have at least 3 components and so $\overline{B}_q \in Y_3$. The composition q can have fewer than 3 components only if $p = 2$ and $q = [n/2, n/2]$. The lemma now follows. $\qquad\square$

Lemma 8 *If n is even and $p = 2$ then*

$$\mathcal{R}(n, p)^2 \subseteq Y_3 \cap \mathcal{T} + Y_4$$

PROOF By Lemma 6 and Lemma 7

$$\mathcal{R}(n,p)^2 \subseteq \langle \overline{B}_{[n/2,n/2]}\rangle\mathcal{R}(n,p) + Y_3 \cap \mathcal{T} + Y_4$$

and so it is sufficient to prove that all products $\overline{B}_{[n/2,n/2]}X$ lie in $Y_3 \cap \mathcal{T} + Y_4$ where X runs through the spanning set of $\mathcal{R}(n,p)$ given in Theorem 2. If $X \in \mathcal{T}$ then, as $\overline{B}_{[n/2,n/2]} \in Y_2$ and $\mathcal{T}$ is a two-sided ideal, $\overline{B}_{[n/2,n/2]}X \in Y_3 \cap \mathcal{T}$.

Suppose that $X = \overline{B}_q$ where $q = [a_1, \ldots, a_r]$ has a repeated part. Then $\overline{B}_{[n/2,n/2]}X$ is a sum of elements $\overline{B}_s$, one for each $2 \times r$ matrix Z in $S([n/2,n/2],q)$. If such a matrix Z has 4 or more non-zero entries then it contributes a summand $\overline{B}_s \in Y_4$. If it has 3 non-zero entries then its two rows will not be equal and it may be paired with the matrix $\bar{Z}$ obtained from Z by interchanging the rows. This pair of matrices contributes a summand $\overline{B}_u + \overline{B}_v$ with $u \approx v$ which lies in $Y_3 \cap \mathcal{T}$. Finally, if Z has 2 non-zero entries only it will have one of two possible forms each of which contributes a summand $\overline{B}_{[n/2,n/2]}$; since $p = 2$ this contribution is zero. □

We can now give the main result of this section.

Theorem 3 *If $n \geq 3$ the nilpotency index of $\mathcal{R}(n,p)$ is $n-1$.*

PROOF In the proof of Corollary 3.5 of [1] it was proved that, if $w = B_{[1,n-1]} - B_{[n-1,1]}$ and $D(a,b) = B_{[1^a,n-a-b,1^b]}$ then

$$w^r = \sum_{k=0}^{r}(-1)^k \binom{r}{k}D(r-k,k)$$

In particular, $w^{n-2} \notin \mathcal{P}_n$ so that $x = \rho_1(w)$ is an element of $\mathcal{R}(n,p)$ and $x^{n-2} \neq 0$. Therefore the nilpotency index of $\mathcal{R}(n,p)$ is not less than $n-1$.

To prove that the nilpotency index is no more than $n-1$ we consider two cases. First, suppose that either n is odd or $p \neq 2$. Then Lemma 6 and Lemma 7 show that

$$\begin{aligned}\mathcal{R}(n,p)^{n-1} &\subseteq (Y_2 \cap \mathcal{T})\mathcal{R}(n,p)^{n-2} + Y_3\mathcal{R}(n,p)^{n-2} \\ &\subseteq Y_n \cap \mathcal{T} + Y_{n+1}\end{aligned}$$

On the other hand, if n is even and $p = 2$, Lemma 6 and Lemma 8 show that

$$\begin{aligned}\mathcal{R}(n,p)^{n-1} &= \mathcal{R}(n,p)^2\mathcal{R}(n,p)^{n-3} \\ &\subseteq (Y_3 \cap \mathcal{T})\mathcal{R}(n,p)^{n-3} + Y_4\mathcal{R}(n,p)^{n-3} \\ &\subseteq Y_n \cap \mathcal{T} + Y_{n+1}\end{aligned}$$

However, since $Y_{n+1} = 0$ and $Y_n \cap \mathcal{T} = 0$, the result now follows. □

Remark By direct calculation we see that $\mathcal{R}(1,p) = 0$ and that $\mathcal{R}(2,p) = \langle \overline{B}_{[1,1]}\rangle$ (so has nilpotency index 2).

References

[1] M.D. ATKINSON, Solomon's descent algebra revisited, *Bull.London Math. Soc.* 24 (1992) 545-551.

[2] C.W.CURTIS and I.REINER, Representation of finite groups and associative algebras, *Interscience Publishers*, New York (1962).

[3] A.M.GARSIA and C.REUTENAUER, A decomposition of Solomon's descent algebra,*Adv. Math.* 77 (1989) 189-262.

[4] I.M. GELFAND, D.KROB, A. LASCOUX, B LECLERC, V. RETAKH and J.-Y. THIBON, Non-commutative symmetric functions, *Adv. Math.* 112 (1995) 218-348.

[5] G. JAMES and A. KERBER, Encyclopedia of mathematics and its applications; v.16. Section:Algebra The representation theory of the symmetric group, Massachusetts, *Addison-Wesley* (1981).

[6] L.SOLOMON, A formula in the group ring of a Coxeter group,*J. Algebra* 41 (1976) 255-268.