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ON ZIGZAG PERMUTATIONS AND COMPARISONS OF ADJACENT ELEMENTS

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Each permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ determines a sequence of " $<$ " and " $>$ " relations determined by the relations holding between adjacent values (a_i, a_{i+1}). A new and elementary algorithm is given, which, for every such pattern of " $<$ ", " $>$ " relations, computes the number of permutations with that pattern. The algorithm enables one to calculate (in bits) the amount of information gained by comparing all adjacent pairs of elements in a list. It also has a simple extension to circular patterns of relations.

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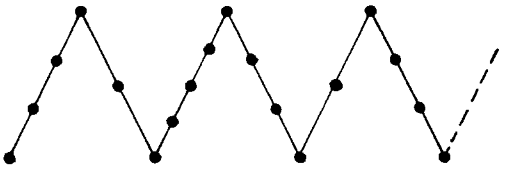
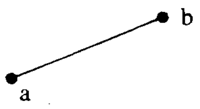
1. Introduction

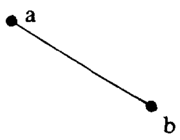
Suppose that $a_1, \ldots, a_n$ are n numerical values and we make the $n-1$ comparisons of a_i against a_{i+1} $(i = 1, 2, \ldots, n-1)$. The results of the comparisons may conveniently be represented by a zigzag diagram



where a link



represents $a < b$ and a link

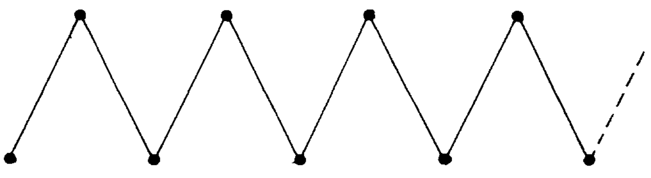


represents $a \geqslant b$. Except in the special cases that all the links go in the same direction, the zigzag shape does not completely determine the ordering of the values. It is therefore interesting to ask how many orderings of the n values could give a particular zigzag diagram; the smaller this number is, the more information is provided by the comparison outcomes as represented by the diagram.

Since only the relative sizes of the values are important for this question, we may consider, without loss of generality, just the case that the n valules are the integers $1, 2, \ldots, n$. The question then becomes: given a zigzag diagram P_n, how many permutations $(a_1, \ldots, a_n)$ of $1, 2, \ldots, n$ are associated with the diagram. Let this number be denoted by $z(P_n)$; the number of information bits given by P_n is then $\log n! - \log z(P_n)$.

If the zigzag is of the simple 'up–down' type

A_n:



we may use an old result of André [1] which states that

$$\sum \frac{z(A_n)x^n}{n!} = \sec x + \tan x.$$

A neat proof of this striking result may be found in [3]. Carlitz [2] gave a formula for $z(P_n)$ for zigzags P_n in which no two 'down' links are consecutive; he remarked that "it is unfortunately rather complicated". Foulkes [4] gave a formula for the general case but, since it requires numbers derived from the representation theory of the symmetric group, it is not easy to use. Instead of a formula, an algorithm for calculating $z(P_n)$ will be given. The algorithm requires about $\frac{1}{2}n^2$ additions, is well suited for both hand and machine calculations, and needs no auxiliary constants. The algorithmic approach also demonstrates the intuitively 'obvious' fact that the up–down pattern of André is the pattern associated with the most permutations. Finally, the number of permutations which are associated with arbitrary fixed circular zigzags (in which there is also an order relation between a_n and a_1) will be discussed.

2. The algorithm

Any zigzag pattern P_n on n nodes is obtained by adding an 'up' link or a 'down' link to a pattern P_{n-1} on $n-1$ nodes. Write $z(P_n, r)$ for the number of permutations associated with P_n with last symbol equal to r (that is, $a_n = r$). The algorithm is based on the following recurrence.

Lemma. (a) *If the last link of* P_n *is an up link, then*

$$z(P_n, r) = \sum_{s=1}^{r-1} z(P_{n-1}, s).$$

(b) *If the last link of* P_n *is a down link, then*

$$z(P_n, r) = \sum_{s=r}^{n-1} z(P_{n-1}, s).$$

Proof. Let $(a_1, \ldots, a_n)$ be any of the $z(P_n, r)$ permutations associated with P_n in which $a_n = r$. In case (a), a_{n-1} is a number s with $1 \leqslant s < r$. For any such s, $(a_1, \ldots, a_{n-1})$ is a permutation of $1, 2, \ldots, r-1, r+1, \ldots, n$, and the correspondence

$$1 \leftrightarrow 1, 2 \leftrightarrow 2, \ldots, r-1 \leftrightarrow r-1,$$
$$r \leftrightarrow r+1, \ldots, n-1 \leftrightarrow n$$

maps it bijectively to a permutation $(b_1, \ldots, b_{n-1})$ of $1, 2, \ldots, n-1$ associated with P_{n-1} and having $b_{n-1} = s$. Thus

$$z(P_n, r) = \sum_{s=1}^{r-1} z(P_{n-1}, s).$$

For case (b), a_{n-1} is a number $s+1$ with $r \leqslant s \leqslant n-1$, and a similar argument applies, but here $a_{n-1} = s+1 > r$ implies $b_{n-1} = s$. □

Since the total number of permutations $z(P_n)$ associated with a pattern P_n is $\sum_r z(P_n, r)$, we obtain the following algorithm to calculate this number. Construct a triangular array similar to Pascal's triangle in that the first row is a single 1 and each subsequent row is formed from the partial sums of the previous row. However, if the *i*th link of P_n is an up link, one forms partial sums in the forward direction, while if it is a down link, one forms partial sums in the backward direction. The final row is then summed to give the result. It follows from the Lemma that the *r*th element of the *n*th row of the triangle is $z(P_n, r)$. The partial summation process can be expressed by the following recurrences:

$$z(P_n, r) = \begin{cases} z(P_n, r-1) + z(P_{n-1}, r-1) \\ \qquad \text{if the final link of } P_n \text{ is up,} \\ z(P_n, r+1) + z(P_{n-1}, r) \\ \qquad \text{if the final link of } P_n \text{ is down.} \end{cases}$$

For the simple up–down pattern of André the triangular array is given in Fig. 1. Its row sums

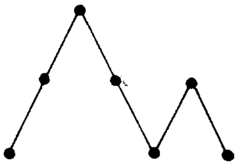
1
0 1
1 1 0
0 1 2 2
5 5 4 2 0
0 5 10 14 16 16
61 61 56 46 32 16 0
....

Fig. 1.

```
                  1
               0     1
            0     0     1
         1     1     1     0
      3     2     1     0     0
   0     3     5     6     6     6
26    26    23    18    12    6     0
```

Fig. 2.

1, 1, 2, 5, 16, 61, 272, ... are the interleaved tangent and secant numbers (and obviously the non-zero numbers on the arms of the triangle are, respectively, the tangent and secant numbers). For the less regular zigzag



we would have the triangular array given in Fig. 2, whose final row sums to 111.

3. Consequences

Each row of a table formed by the rules above is an increasing sequence either when read from left to right or when read from right to left. The next row will also be increasing in one of these directions since it consists of the partial sums of the current row in either the increasing or decreasing directions. Notice that, for the up–down pattern of André, the summations are always in the decreasing direction. Since partial summation in the decreasing direction produces a sequence with larger terms than partial summation in the increasing direction, it follows that, for any n, $z(P_n)$ is largest for André's pattern.

The algorithm can be extended to circular zigzag patterns where the ordering between a_n and a_1 is prescribed. In any permutation of $1,\ldots,n$ associated with a circular zigzag pattern the symbol n must be attached to one of the local maxima. Then, clearly, the number of permutations associated with the pattern which have n attached to a certain local maximum will be equal to the number of permutations of $1,\ldots,n-1$ associated with the ordinary zigzag pattern obtained by deleting the two edges of the circular pattern incident with this local maximum. Hence, to find the number of permutations associated with a circular zigzag pattern it suffices to compute $\Sigma\, z(P_{n-1})$ (where the summation is over all patterns P_{n-1} obtained by deleting a local maximum and its incident edges from the circular zigzag pattern).

As an example, consider the permutations of $1,\ldots,8$ satisfying

$$a_1 < a_2 < a_3 > a_4 < a_5 > a_6 < a_7 < a_8 > a_1.$$

This pattern has local maxima at a_3, a_5, and a_8 and hence the numbers of permutations associated with the following three patterns must be calculated:

$$x_1 < x_2 > x_3 < x_4 < x_5 > x_6 < x_7,$$

$$x_1 < x_2 < x_3 > x_4 < x_5 < x_6 > x_7,$$

$$x_1 < x_2 < x_3 > x_4 < x_5 > x_6 < x_7.$$

By following the algorithm in the previous section these numbers are easily computed as 169, 99, 155. Their sum, 423, is the number of permutations associated with the given circular pattern.

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