The p-Modular Descent Algebras

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Abstract

Solomon's descent algebras are studied over fields of prime characteristic. Their radical and irreducible modules are determined. It is shown how their representation theory can be related to the representation theory in fields of characteristic zero.

1 Introduction

Descent algebras are non-commutative, non-semi-simple algebras associated with finite Coxeter groups. They were first discovered by Solomon in the 1970's and for the last 10 years have been studied intensively [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 17]. Previous work has concentrated on the case that the underlying field has characteristic zero. In this paper we study descent algebras over fields of prime characteristic. We determine the radical of a descent algebra in characteristic p, and its irreducible modules. Once the irreducible modules have been found we turn our attention to the decomposition theory and give a complete account for all descent algebras and all characteristics.

Let W be a Coxeter group with generating set S of fundamental reflections. If L is any subset of S let W_L be the subgroup generated by L. W_L is called a *standard parabolic* subgroup of W and any subgroup conjugate to a standard parabolic subgroup is said to be parabolic. Let X_L be the (unique) set of minimal length representatives of the left cosets of W_L in W. Notice that $X_L^{-1} = \{g^{-1} | g \in X_L\}$ is then a set of representatives (also of minimal length)

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for the right cosets of W_L and that $X_K^{-1} \cap X_L$ is a set representatives for the double cosets corresponding to W_K, W_L.

Solomon proved the following remarkable theorem:

Theorem 1 *[22] For every subset K of S let*

$$x_K = \sum_{w \in X_K} w.$$

Then

$$x_J x_K = \sum a_{JKL} x_L$$

where a_{JKL} is the number of elements $g \in X_J^{-1} \cap X_K$ such that $g^{-1} W_J g \cap W_K = W_L$ with $L = g^{-1} J g \cap K$.

The set of all x_K is therefore a basis for an algebra Σ_W over the field of rationals with integer structure constants a_{JKL}. This algebra is now known as the *descent algebra* of W and much is known about its structure.

Solomon himself began the study of this algebra by determining its radical, $\mathrm{rad}(\Sigma_W)$, and some properties of $\Sigma_W/\mathrm{rad}(\Sigma_W)$. To describe his results let χ_K be the permutation character of W acting on the right cosets of W_K and let G_W be the Z-module generated by all χ_K. Note that each generalised character in G_W has integer values on the elements of W.

Theorem 2 *[22]*

1. $\mathrm{rad}(\Sigma_W)$ *is spanned by all differences $x_J - x_K$ where J and K are conjugate subsets of S*
2. *the linear map θ defined by the images $\theta(x_K) = \chi_K$ is an algebra homomorphism, and* $\ker\theta = \mathrm{rad}(\Sigma_W)$

Since the structure constants a_{JKL} are integers the Z-module $\mathcal{Z}_W$ spanned by all x_K is a subring (an order) of Σ_W. This allows us to study the p-modular version of the descent algebra for any prime p. For any prime p, $p\mathcal{Z}_W$ is an ideal of $\mathcal{Z}_W$. We define $\Sigma(W,p) = \mathcal{Z}_W / p\mathcal{Z}_W$, the p-modular descent algebra of W. Obviously, $\Sigma(W,p)$ is an algebra over $\mathcal{F}_p$, the field of order p.

Let ρ_1 be the natural projection $\mathcal{Z}_W \to \Sigma(W,p)$ and let $\overline{x}_J = \rho_1(x_J)$. Then

$$\overline{x}_J \overline{x}_K = \sum \overline{a}_{JKL} \overline{x}_L$$

where $\overline{a}_{JKL}$ is the image of a_{JKL} in $\mathcal{F}_p$. Furthermore let ρ_2 be the map defined on G_W which reduces character values modulo p, and let $G(W,p)$ be the image of ρ_2.

The map $\phi : \Sigma(W,p) \to G(W,p)$ defined by

$$\phi(\rho_1(x)) = \rho_2(\theta(x)) \text{ for all } x \in \mathcal{Z}_W$$

is clearly well-defined and is an algebra homomorphism. In section 3 we shall give an analogue of Solomon's theorem to describe the radical of $\Sigma(W,p)$ using the homomorphism ϕ. In section 4 we build on this result by defining the irreducible modules of $\Sigma(W,p)$. Then we relate the representation theory of $\Sigma(W,p)$ to that of Σ_W. Our main result is a complete description of the decomposition matrix for every descent algebra Σ_W and prime p. Some of this theory is generic, applying to every descent algebra, but the fine details require the classification of Coxeter groups and a case by case analysis.

2 The parabolic table of marks

In this section we recall some preliminary results that we shall need later.

Let W be a Coxeter group with generating set of fundamental reflections S and let $W_1(=1), W_2, \ldots, W_r(=W)$ be representatives of the conjugacy classes of subgroups of W.

The *table of marks* of W is the $r \times r$-matrix

$$M(W) = \left(\left|\mathrm{Fix}_{W/W_i}(W_j)\right|\right)_{i,j=1,\ldots,r}$$

which records for subgroups W_i, W_j of W the number of fixed points of W_j in the action of W on the cosets of W_i (the *mark* of W_j on W/W_i), and where both W_i and W_j run through the representatives of conjugacy classes of subgroups of W.

We have

$$\left|\mathrm{Fix}_{W/W_i}(W_j)\right| = [N_W(W_i) : W_i] \cdot |\{W_i^x \mid x \in W,\, W_j \leq W_i^x\}|$$

Thus, with a suitable ordering of the representatives W_i, we see that $M(W)$ is a lower triangular matrix with non-zero entries on the diagonal, and therefore invertible.

The *parabolic table of marks* is a certain principal submatrix $M^c(W)$ of $M(W)$. This submatrix is defined by those rows and columns of $M(W)$ that correspond to the (standard) parabolic subgroups of W. It is known [7] that the set of conjugacy classes of parabolic subgroups is in one-to-one correspondence with the set E of conjugacy classes of subsets of S via the mapping $J \mapsto W_J = \langle J \rangle$. Therefore we have

$$M^c(W) = \left(\left|\mathrm{Fix}_{W/W_J}(W_K)\right|\right)_{J,K \in E}$$

where we also write $\beta_{JK} = \left|\mathrm{Fix}_{W/W_J}(W_K)\right|$ for any $J, K \subseteq S$.

Letting c_K be a Coxeter element of W_K and recalling that χ_J is the permutation character of W on W/W_J we have the following result [7].

Lemma 1

$$\begin{aligned}\beta_{JK} &= [N_W(W_J) : W_J] \cdot |\{W_J^w \mid w \in W,\, W_K \leq W_J^w\}| \\ &= \left|\{w \in X_J^{-1} \cap X_K \mid J^w \cap K = K\}\right| \\ &= a_{JKK} \\ &= \chi_J(c_K.\end{aligned}$$

In particular, $\beta_{JJ} = [N_W(W_J) : W_J] \neq 0$ *and* β_{JJ} *divides* β_{JK} *for every* $K \subseteq S$.

3 The radical of $\Sigma(W, p)$

The main aim of this section is to prove the following p-modular analogue of Theorem 2.

Theorem 3 $\mathrm{rad}(\Sigma(W,p)) = \ker\phi$. *Moreover,* $\mathrm{rad}(\Sigma(W,p))$ *is spanned by all* $\overline{x}_J - \overline{x}_K$ *where* J, K *are conjugate subsets of* S*, together with all* $\overline{x}_J$ *for which* p *divides* $[N_W(W_J) : W_J]$.

Let r be the number of rows of $M^c(W)$ and let s be the number of rows indexed by subsets J with $p \not| [N_W(W_J) : W_J]$.

Lemma 2 1. *$M^c(W)$ is a lower triangular matrix of rank $r = \dim G_W$.*

2. *The p-rank of $M^c(W)$ (i.e. the rank of $M^c(W)$ modulo p or $\dim G(W,p)$) is s.*

PROOF The first part follows from Section 2. If p divides a diagonal entry of $M^c(W)$ then, by Lemma 1, p divides every entry of that row. Thus the rank of $M^c(W) \mod p$ (i.e. $\dim G(W,p)$) is the number of non-zero rows in $M^c(W) \mod p$ and this, by Lemma 1 again, is s.

Lemma 3 1. *$\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ is commutative.*

2. *Every nilpotent element of $\Sigma(W,p)$ lies in $\mathrm{rad}(\Sigma(W,p))$.*

PROOF Let θ_1 be the restriction of θ to $\mathcal{Z}_W$. Then θ_1 maps $\mathcal{Z}_W$ onto the commutative ring G_W. By Theorem 2, the kernel of θ_1 is the Z-module $\mathcal{R}_W$ spanned by all $x_J - x_K$ where J and K are conjugate subsets of S, and is a nilpotent ideal of $\mathcal{Z}_W$. In particular $\rho_1(\mathcal{R}_W)$ is a nilpotent ideal of $\Sigma(W,p)$, and therefore $\rho_1(\mathcal{R}_W) \subseteq \mathrm{rad}(\Sigma(W,p))$. Hence there exists an ideal $\mathcal{S}_W$ of Σ_W, the pre-image of $\mathrm{rad}(\Sigma(W,p))$, such that $\mathcal{R}_W \subseteq \mathcal{S}_W$ and $\mathcal{S}_W/p\mathcal{Z}_W \cong \mathrm{rad}(\Sigma(W,p))$. Since $\Sigma(W,p) \cong \mathcal{Z}_W/p\mathcal{Z}_W$, $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p)) \cong \mathcal{Z}_W/\mathcal{S}_W$ is a homomorphic image of $\mathcal{Z}_W/\mathcal{R}_W \cong G_W$. Since the latter ring is commutative the first part follows.

If x is any nilpotent element of $\Sigma(W,p)$ then the coset $x + \mathrm{rad}(\Sigma(W,p))$ is a nilpotent element in the commutative semi-simple algebra $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ and so is zero. Therefore $x \in \mathrm{rad}(\Sigma(W,p))$ proving the second part.

Proof of Theorem 3.

First we note that $\mathrm{rad}(\Sigma(W,p)) \subseteq \ker\phi$. This is because the image of ϕ is an algebra of functions defined over a field and is therefore semi-simple. Consequently the two-sided nilpotent ideal $\phi(\mathrm{rad}(\Sigma(W,p)))$ must be zero.

Now we prove that, if $p|[N_W(W_J) : W_J]$, then $\overline{x}_J \in \mathrm{rad}(\Sigma(W,p))$. From the definition of a_{JKL} in Theorem 1, $\overline{a}_{JKL} = 0$ unless $L \subseteq K$ and, by Lemma 1, $\overline{a}_{JKK} = 0$ also. Thus $\overline{x}_J\overline{x}_K$ is a linear combination of elements $\overline{x}_L$ with $L \subset K$ (and so $|L| \leq |K| - 1$). Now, by induction, it follows that $\overline{x}_J^t\overline{x}_K$ is a linear combination of elements $\overline{x}_L$ with $|L| \leq |K| - t$ and so $\overline{x}_J^{|K|+1}\overline{x}_K=0$ for all K. In particular $\overline{x}_J$ is nilpotent and so $\overline{x}_J \in \mathrm{rad}(\Sigma(W,p))$ by Lemma 3.

The elements $\overline{x}_J - \overline{x}_K$ where J and K are conjugate subsets of S are all nilpotent and, by Lemma 3, lie in $\mathrm{rad}(\Sigma(W,p))$. They span a space U of dimension $\dim\mathrm{rad}(\Sigma_W) = \dim\Sigma_W - \dim G_W = 2^{n-1} - r$. In addition there are $r - s$ elements $\overline{x}_J$ corresponding to those rows of $M^c(W)$ for which $p|[N_W(W_J) : W_J]$ which also lie in $\mathrm{rad}(\Sigma(W,p))$. These, together with U, span a space of dimension $2^{n-1} - r + (r - s) = 2^{n-1} - \dim G(W,p) = \dim\ker\phi$. Hence $\dim\mathrm{rad}(\Sigma(W,p)) \geq \dim\ker\phi$.

This proves that $\ker\phi = \mathrm{rad}(\Sigma(W,p))$ as required and that it is spanned by the desired set of elements.

4 Representation Theory of $\Sigma(W,p)$

The representation theory of Σ_W has not been much studied in general although some results for the Coxeter groups of types A and B have been found [2, 5, 6, 14]. In this section we show how the representation theory of $\Sigma(W,p)$ depends on that of Σ_W. Specifically, we shall be interested in the composition factors of the principal indecomposable modules (indecomposable summands of the regular module) for each of Σ_W and $\Sigma(W,p)$. The first observation is straightforward: a representation of Σ_W over $\mathcal{F}_p$ necessarily has $p\mathcal{Z}_W$ in its kernel and so induces a representation of $\Sigma(W,p)$; moreover, every representation of $\Sigma(W,p)$ arises in this way. Therefore we may study the representation theory of $\Sigma(W,p)$ by examining the p-modular representations of Σ_W. We do this in the manner pioneered in group theory: by relating the representations in characteristic zero to those in characteristic p via a decomposition matrix.

This approach is tractable because the irreducible representations are all 1-dimensional. In fact, since $\Sigma_W/\mathrm{rad}(\Sigma_W)$ and $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ are commutative of dimensions r and s respectively (where r and s have the meanings given in the previous section) Σ_W has r 1-dimensional irreducible representations over a field of characteristic zero and s 1-dimensional irreducible representations over a field of characteristic p. It follows (see 54.16, [12]) that the multiplicities of the principal indecomposable modules as direct summands in the regular representation of both Σ_W and $\Sigma(W,p)$ are all 1.

We can explicitly describe the irreducible representations. As in Section 2 let E denote the set of representatives of the subsets of S that index the rows and columns of $M^c(W)$. For each $K \in E$ define the map $\lambda_K : \Sigma_W \to$ by

$$\lambda_K(x) = \theta(x)(c_K) \text{ for all } x \in \Sigma_W$$

Since θ is a homomorphism it follows readily that λ_K is also a homomorphism, therefore a 1-dimensional representation of Σ_W. Notice that λ_K is completely determined by its values on basis elements x_J, that $\lambda_K(x_J) = \theta(x_J)(c_K) = \chi_J(c_K) = \beta_{JK}$, and these values of λ_K comprise the column of the matrix $M^c(W)$ indexed by K. In particular, $\lambda_K|_{\mathcal{Z}_W}$ takes integer values and reducing these values modulo p we shall obtain the irreducible representations in a field of characteristic p. We already knew (Lemma 2) that the p-rank of $M^c(W)$ was s and so the above arguments have now proved:

Lemma 4 *1. The columns of $M^c(W)$ define the irreducible representations of Σ_W, and*

2. The columns of $M^c(W)$ modulo p define the irreducible representations of $\Sigma(W,p)$ and $M^c(W)$ modulo p has precisely s distinct columns.

According to this lemma the set E indexes the irreducible representations of Σ_W. We now select a subset $F \subseteq E$ to index the irreducible representations of $\Sigma(W,p)$. In principle any subset that indexes s distinct columns of $M^c(W)$ mod p will suffice but we shall make a specific choice so that our results are easier to state. In $M^c(W) \mod p$ there are exactly s non-zero rows (see the proof of Lemma 4) and we let $F \subseteq E$ index this set of rows. Since $M^c(W)$ mod p is lower triangular of rank p, F also indexes a set of distinct columns of $M^c(W) \mod p$. We define a matrix $D = (d_{KL})$ whose rows and columns are indexed by the members of E and F respectively. If $K \in E, L \in F$ then $d_{KL} = 1$ if columns K and L of $M^c(W)$ are equal modulo p, $d_{KL} = 0$ otherwise. By the previous lemma, the sets E and F index the irreducible representations of Σ_W and $\Sigma(W,p)$ respectively and, since D determines the structure of each irreducible representation of Σ_W when reduced modulo p, we have

Proposition 1 *D is the decomposition matrix of the algebra Σ_W.*

Moreover we have the following easy result:

Proposition 2 *Let $K \in E, L \in F$ head columns of the matrix $M^c(W)$. Then, if c_K and c_L have conjugate p-regular parts, $d_{KL} = 1$.*

PROOF. By the arguments in §82 of [12] every character χ_J takes equal values modulo p on c_K and c_L. Thus, $\lambda_K = \lambda_L \mod p$ and so $d_{KL} = 1$.

In the remainder of this section we shall consider descent algebras according to their Coxeter type. By a combination of theoretical argument and computer calculation we obtain a description of the decomposition matrix in all cases and this shows that, often, the converse of Proposition 2 is true.

We let $\pi(n)$ denote the number of partitions of n. This non-standard notation is necessary since we also define $\pi(n, p)$ as the number of partitions of n in which no part has multiplicity p or more. We note the following result.

Lemma 5 *$\pi(n, p)$ is the number of partitions of n into parts not divisible by p.*

This lemma is proved combinatorially in [19] (p.41). It has an algebraic proof also since the two expressions for $\pi(n, p)$ are the number of p-regular conjugacy classes of the symmetric group S_n. One of them naturally parametrizes such conjugacy classes; the other naturally parametrizes absolutely irreducible modules for S_n over fields of characteristic p.

4.1 Representation Theory of $\Sigma(A_{n-1}, p)$

In this subsection we let $W = A_{n-1}$ which is best described as the symmetric group S_n acting in the usual way on $\{1, 2, \ldots, n\}$ with generating set $S = \{(i, i+1) | i = 1, \ldots, n-1\}$. If $K \subseteq S$ then the Coxeter element c_K has cycles on sets $[u..v]$ of consecutive integers. The ordered list of cycle lengths (one cycle appearing before another if it permutes integers with smaller values) determines and is determined by K. Therefore the subsets of S can be parameterised by compositions of n. The following lemma and corollary are easy consequences of this parameterisation and Lemmas 2 and 4

Lemma 6 *1. If $K, L \subseteq S$ then K is conjugate to L if and only if the corresponding compositions determine the same partition of n.*

2. If $K \subseteq S$ and its corresponding composition has a_i components equal to i (for $i = 1, \ldots, n$) then

$$[N(W_K) : W_K] = a_1!\, a_2! \cdots a_n!$$

Corollary 1 *1. $r = \pi(n)$*

2. $s = \pi(n,p)$

Theorem 4 *Let W be one of the Coxeter groups A_{n-1} and let $K \in E$, $L \in F$. Then $d_{KL} = 1$ if and only if c_K and c_L have conjugate p-regular parts.*

Proof There are two equivalence relations δ_1, δ_2 on the set E (which indexes the columns of $M^c(W)$):

$$(K,J) \in \delta_1 \text{ if } \lambda_K = \lambda_J \mod p$$

$$(K,J) \in \delta_2 \text{ if } c_K, c_J \text{ have conjugate } p\text{-regular parts}$$

We have seen (Proposition 2) that $\delta_2 \subseteq \delta_1$. However, the number of δ_1-equivalence classes is s (Lemma 4) and this is $\pi(n,p)$ (Corollary 1). By Lemma 5 this is also the number of partitions with no part divisible by p which is the number of equivalence classes of δ_2. Hence $\delta_1 = \delta_2$ and the theorem follows.

4.2 Representation Theory of $\Sigma(B_n, p)$

It is convenient to represent B_n as a permutation group on $\{\pm 1, \ldots, \pm n\}$ with block system $\{i, -i\}_{i=1}^n$ on which it acts as the full symmetric group with kernel of order 2^n. The set of Coxeter generators $S = \{s_0, s_1, \ldots, s_{n-1}\}$ is defined as $s_0 = (-1, 1)$ and $s_i = (i, i+1)(-i, -i-1), 1 \leq i \leq n-1$.

Let $K \subseteq S$ and consider the Coxeter element c_K. If c_K has a cycle $(a, b, \ldots)$ consisting of positive elements (a positive cycle) then it will also have a corresponding negative cycle $(-a, -b, \ldots)$. Furthermore, at most one cycle of c_K can contain both positive and negative elements; such a cycle is present if and only if $s_0 \in K$. We may write

$$c_K = x_0 x_1 \tag{1}$$

where x_0 is the cycle containing both positive and negative elements (or $x_0 = 1$ if there is no such cycle) and x_1 is the product of all the other cycles (positive and negative in matching pairs); note that x_0 commutes with x_1. Each positive cycle is on some range $[u..v]$ of consecutive integers and the list of lengths of positive cycles taken in the natural order (as in the previous subsection) determines and is determined by K. In this way the subsets of S can be parameterised by compositions of integers $m, 0 \leq m \leq n$. The following result is a consequence of the results of [18].

Lemma 7 *1. If $K, L \subseteq S$ then K is conjugate to L if and only if the corresponding compositions determine the same partition.*

2. If $K \subseteq S$ and the corresponding composition is a composition of m with a_i components of size i and t components in all then

$$[N(W_K) : W_K] = 2^t a_1!\, a_2! \cdots a_m!$$

Corollary 2 *1.* $r = \sum_{m=0}^{n} \pi(m)$

2. If $p \neq 2$ *then* $s = \sum_{m=0}^{n} \pi(m,p)$

Let K be one of the subsets indexing the rows and columns of $M^c(W)$ and $c_K = x_0 x_1$ as in Equation 1. If x_1 is a p-regular element we say that K is a *p-special* subset of S. Since the order of x_1 is the lowest common multiple of its cycle lengths, K is p-special if and only if the partition corresponding to K has no part divisible by p. By Lemma 5 and Corollary 2, there are precisely s p-special subsets when $p \neq 2$.

Lemma 8 *If* $K \subseteq S$ *there exists a p-special* $K_1 \subseteq S$ *such that* c_K *and* c_{K_1} *have conjugate p-regular parts.*

Proof Let $c_K = x_0 x_1$ as in Equation 1 and let x_2 be the p-regular part of x_1. Since x_2 is a power of x_1, its cycles also come in matching positive, negative pairs. Therefore x_2 is conjugate, via a permutation in the centraliser of x_0, to a Coxeter element x_3 with the same properties as x_2. But then $x_0 x_3$ is also a Coxeter element c_{K_1} whose p-regular part is conjugate to that of $x_0 x_1$.

Lemma 9 *If* $p \neq 2$ *the columns of* $M^c(W)$ *which are indexed by the p-special subsets provide a full set of irreducible representations of* $\Sigma(B_n, p)$.

Proof By the last lemma the columns of $M^c(W) \mod p$ indexed by p-special subsets contain a full set of distinct columns and since there are s such columns they must yield a complete set of irreducible representations of $\Sigma(B_n, p)$.

Theorem 5 *Let* W *be one of the Coxeter groups* B_n *and let* $K \in E, L \in F$*. If* $p \neq 2$ *then* $d_{KL} = 1$ *if and only if* c_K *and* c_L *have conjugate p-regular parts. If* $p = 2$ *then* $F = \{S\}$ *and* $d_{KS} = 1$ *for all* K.

Proof Suppose first that $p \neq 2$. Proposition 2 has proved one implication already. For the other, suppose $d_{KL} = 1$ and let K_1, L_1 be the p-special subsets, guaranteed by Lemma 8, such that K, K_1 have conjugate p-regular parts and L, L_1 have conjugate p-regular parts. Then, by Proposition 2, $d_{K_1 L_1} = 1$ and Lemma 9 shows that $K_1 = L_1$.

If $p = 2$, Lemma 7 implies that the only $K \in E$ for which 2 does not divide $[N(W_K) : W_K]$ is the one with $t = 0$, namely $K = S$. Therefore $\Sigma(B_n, 2)$ has just one irreducible representation and so $d_{KL} = 1$ for all $K \in E$, $L \in F = \{S\}$.

4.3 Representation Theory of $\Sigma(D_n, p)$.

The Coxeter group (W, S) of type D_n can be considered as a normal subgroup of index 2 in the Coxeter group $(\hat{W}, \hat{S})$ of type B_n. As such its set of Coxeter generators is $S = \{u, s_1, \ldots, s_{n-1}\}$ where, as in the previous subsection, $\hat{S} = \{s_0, s_1, \ldots, s_{n-1}\}$ and $u = s_0 s_1 s_0 = (-1,1)(1,2)(-1,-2)(-1,1) = (-1,2)(1,-2)$.

For any $K \subseteq S$ the parabolic subgroup W_K is isomorphic to $W_0 \times W_1$ where W_0 is of type D_{n_0} for some $n_0 \leq n$, $n_0 \neq 1$ and $W_1 = \langle K_1 \rangle$ for some $K_1 \subseteq \{s_{n_0}, \ldots, s_{n-1}\}$. (Here the group of type D_2 is $\langle u, s_1 \rangle$ and isomorphic to a group of type $A_1 \times A_1$ and the group of type D_3 is $\langle u, s_1, s_2 \rangle$ and isomorphic to a group of type A_3.) If $n_0 = 0$ then W_1 is a subgroup of either the group W' generated by $S' = \{s_1, s_2, \ldots, s_{n-1}\}$ or the group W'' generated by $S'' = \{u, s_2, s_3, \ldots, s_{n-1}\}$ which are both of type A_{n-1}.

Thus to each subset $K \subseteq S$ there is associated via W_1 a composition of $m \leq n$. Each composition occurs this way, except those of $n-1$. Conversely, for each composition λ of $m \neq n-1$, there is a unique $K \subseteq S$, unless λ is a composition of n with $\lambda_1 > 1$. In that case there are two subsets with that label, each containing exactly one of s_1 and u.

Consider $K, L \subseteq S$. Then K and L are conjugate in W if and only if their corresponding compositions determine the same partition, unless this partition is a partition of n with all parts even. In that case K and L are conjugate only if they both lie in S' or both in S''.

Consider $W_K = W_0 \times W_1$ with W_0 of type D_{n_0} for some $n_0 \geq 2$. Then there is a parabolic subgroup $\hat{W}_K = \hat{W}_0 \times W_1$ of $\hat{W}$ where $\hat{W}_0$ is of type B_{n_0}. We have $W_K = \hat{W}_K \cap W$ and $[\hat{W}_K : W_K] = 2$. Also $[N_{\hat{W}}(\hat{W}_K) : N_W(W_K)] = 2$ whence β_{KK} is computed from the partition corresponding to K in the same way as in case B_n.

Now let $n_0 = 0$ and let W_K be a subgroup of W' with corresponding partition μ. Then W_K is a parabolic subgroup of both W and $\hat{W}$. We have $N_{\hat{W}}(W_K) \subseteq W$ if and only if all parts of μ are even. We thus get the following formula.

Lemma 10 *Let $K \subseteq S$ with corresponding partition $\mu = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$. Then*

$$[N_W(W_K) : W_K] = 2^{m_1} m_1! \cdots 2^{m_n} m_n! \, a$$

where $a = 1$ unless μ is a partition of n and has at least one odd part. In that case $a = 1/2$.

Let c_K be a Coxeter element of W_K. Again, we have a unique decomposition $c_K = x_0 x_1$ where $x_0 \in W_0$ and $x_1 \in W_1$. We call K a p-special subset if x_1 is p-regular. And, by the same argument as for type B_n, we have that for each $K \subseteq S$ there is a p-special $K_1 \subseteq S$ such that c_K and c_{K_1} have conjugate p-regular parts.

Similar considerations as for type B_n then lead to the following description of the decomposition matrix for type D_n.

Theorem 6 *Let (W,S) be of type D_n and let $K \in E$, $L \in F$. If $p \neq 2$ then $d_{KL} = 1$ if and only if c_K and c_L have conjugate p-regular parts. If $p = 2$ and n is even then we have $F = \{S\}$ and $d_{KS} = 1$ for all $K \in E$; if n is odd then we have $F = \{S', S\}$ and $d_{KL} = 1$ if and only if either $L = S$ and $K \neq S'$ or $L = S'$ and $K = S'$.*

PROOF The theorem for $p \neq 2$ follows as in case B_n and we now suppose that $p = 2$. Then, by Lemma 10, $\beta_{LL} = [N_W(W_L) : W_L]$ is odd only if all $m_i = 0$ (whence μ is the empty partition corresponding to $L = S$) or, if μ is a partition of n with at least one odd part, at most one $m_i = 1$, and all other $m_i = 0$ (whence n is odd and μ is the partition $[n]$ corresponding to $L = S'$). Thus, if n is even, $F = \{S\}$ and $d_{KL} = 1$ for all $K \in E$.

If n is odd then $F = \{S, S'\}$. Since $\beta_{S'S} = 0$ and $\beta_{S'S'} = 1$ we shall have $d_{KS} = 1$ or $d_{KS'} = 1$ according as $\beta_{S'K}$ is even or odd.

To resolve this case we consider the following action on complementary pairs, first as a B_n action. Let $I = \{1, \ldots, n\}$ and let

$$X = \big\{\{P, Q\} \bigm| P, Q \subseteq I; P \cup Q = I; P \cap Q = \emptyset\big\}$$

(so we always have $Q = I \setminus P$). Then B_n acts on X as follows. The action of s_i $(i \geq 1)$ is induced from its action as $(i, i+1)$ on I and the action of s_0 is given by

$$\{P, Q\}^{s_0} = \{P \perp \{1\}, Q \perp \{1\}\},$$

where $A \perp B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets A, B. Note that, if we define $t_i = s_i \cdots s_1 s_0 s_1 \cdots s_i$ then t_i acts as

$$\{P, Q\}^{t_i} = \{P \perp \{i{+}1\}, Q \perp \{i{+}1\}\},$$

the symmetric difference with $\{i{+}1\}$, and the longest element $w_0 = t_0 t_1 \cdots t_{n-1}$ of $\hat{W}$ acts as symmetric difference with I whence it fixes every point in X. The complementary pair $\{P, Q\}$ arises from $\{\emptyset, I\}$ by taking symmetric differences with P (or Q). Thus the action of $\hat{W}$ is transitive on all of the 2^{n-1} complementary pairs in X and the stabiliser of $\{\emptyset, I\}$ is $\langle s_1, \ldots, s_{n-1}, w_0\rangle$, a group of index 2^{n-1} in $\hat{W}$.

Now restrict the action to W. Then, since in the case of odd n we have $w_0 \notin W$, the stabiliser of $\{\emptyset, I\}$ in W is W', which is of index 2^{n-1} in W. Hence W acts transitively on X and the action is equivalent to the action on the cosets of W'. Therefore we can use it to determine the values $\beta_{S'K}$.

We know that $\beta_{S'K} = 0$ whenever W_K is not conjugate to a subgroup of $W_{S'} = W'$. It remains to investigate the fixed points of parabolic subgroups of W' which is of type A_{n-1}. Consider s_1 and its fixed points. If $n > 2$ then $\{P, Q\}$ is stable under s_1 if and only if $\{1, 2\} \subseteq P$ or $\{1, 2\} \subseteq Q$. In either case taking symmetric differences with $\{1,2\}$ yields a different point $\{P', Q'\}$ which is also fixed by s_1. So the fixed points of s_1 come in pairs.

A similar argument applies to a Coxeter element c_K of any parabolic subgroup W_K of W' unless $K = S'$. Here we denote by $J \subseteq I$ the set of points moved by c_K. Then we find that $\{P, Q\}$ is stable under c_K if and only if $J \subseteq P$ or $J \subseteq Q$. Again, taking symmetric differences with J produces a different fixed point $\{P', Q'\}$. This shows that $\beta_{S'K}$ is even for all proper parabolic subgroups W_K of W'.

4.4 Representation Theory of Exceptional Types

The descriptions of the decomposition matrices in the case of the classical types in the previous subsections are special cases of a more general classification of columns of the parabolic table of marks that are equal if taken mod p.

For this more general classification we need to extend the notion of "having the same p-regular part". Let w' be the p-regular part of $w \in W$ and let $\to_p$ be the relation on E defined by $J \to_p K$ if $\langle w' \rangle^c$ is conjugate to W_K and $\langle w \rangle^c$ is conjugate to W_J.

Theorem 7 *Let $K \in E$ and $L \in F$. Then $d_{KL} = 1$ if and only if K and L lie in the same class of the equivalence generated by $\to_p$.*

The following tables 1–6, which we have computed using the CHEVIE [15] package in GAP [21], describe the decomposition matrices for the exceptional types. The proof of the theorem follows by inspection of these tables and the parabolic tables of marks reduced mod p, together with Theorems 4, 5 and 6. Note that the theorem is also true for the dihedral types $I_2(m)$ (see [23] for a full account of the representation theory in all characteristics in this case).

In each case we give for any $K \in E$ and for any prime p dividing the order of W the list of $L \in E$ such that $K \to_p L$. The first entry in each list is determined by the p-regular part of a Coxeter element, and the number in parenthesis denotes the representative in the equivalence obtained as the closure of the relation $\to_p$ if different from the first entry of the list. If the list for K consists of K only and this is also the representative we just have a dot (.) as the entry. Note that conversely, all the representatives have a dot as the entry.

For each $K \in E$ we also list its isomorphism type, possibly with dashes ($'$ and $''$) to distinguish isomorphic parabolic subgroups, and the index β_{KK} of W_K in its normalizer in W.

		β_{KK}	$p=2$	$p=3$	$p=5$
1	1	51840	.	.	.
2	A_1	720	1	.	.
3	$A_1 \times A_1$	48	1	.	.
4	A_2	72	4 (1)	1	.
5	$A_1 \times A_1 \times A_1$	12	1	.	.
6	$A_2 \times A_1$	6	4 (1)	2	.
7	A_3	8	1	.	.
8	$A_2 \times A_1 \times A_1$	2	4 (1)	3	.
9	$A_2 \times A_2$	12	.	1	.
10	$A_3 \times A_1$	2	1	.	.
11	A_4	2	.	.	1
12	D_4	6	4, 1 (1)	12 (1)	.
13	$A_2 \times A_2 \times A_1$	2	9	2	.
14	$A_4 \times A_1$	1	11	.	2
15	A_5	2	9	5	.
16	D_5	1	1, 4	.	.
17	E_6	1	17, 9 (9)	12, 1 (1)	.

Table 1: Decomposition matrix for E_6.

4.5 Cartan matrices

The representation theory of $\Sigma(W,p)$ can now be studied using that of Σ_W. As a first step in this direction we consider the Cartan invariants. First we recall their definition:

Definition 1 *Let A be any finite dimensional algebra and let $P_1, \ldots, P_r$ be a complete set of principal indecomposable modules for A, with corresponding irreducible modules $T_1, \ldots, T_r$. Let c_{ij} be the multiplicity of T_j as a composition factor of P_i. Then the $r \times r$ matrix $C = [c_{ij}]$ is called the Cartan matrix of A.*

As a special case of the results of [16] we have

Theorem 8 *Let C and $\tilde{C}$ be the Cartan matrices of the descent algebra Σ_W and its p-modular counterpart $\Sigma(W,p)$ and let D be the decomposition matrix of Σ_W. Then $\tilde{C} = D^T C D$.*

By Theorems 8 and 7 we immediately see that the Cartan matrix of $\Sigma(W,p)$ can be determined once it is known for Σ_W. Types A and B can therefore be handled by Theorem 5.4 of [14] and Theorem 3.3 of [6] which give the Cartan matrices in characteristic zero. Furthermore, the work of [23] allows the dihedral case to be solved. However, we have not calculated the Cartan matrix in characteristic

		β_{KK}	$p=2$	$p=3$	$p=5$	$p=7$
1	1	2903040	.	.	.	.
2	A_1	23040	1	.	.	.
3	$A_1\times A_1$	768	1	.	.	.
4	A_2	1440	4 (1)	1	.	.
5	$(A_1\times A_1\times A_1)'$	1152	1	.	.	.
6	$(A_1\times A_1\times A_1)''$	96	1	.	.	.
7	$A_2\times A_1$	48	4 (1)	2	.	.
8	A_3	96	1	.	.	.
9	$A_1\times A_1\times A_1\times A_1$	48	1	.	.	.
10	$A_2\times A_1\times A_1$	8	4 (1)	3	.	.
11	$A_2\times A_2$	24	11 (1)	1	.	.
12	$(A_3\times A_1)'$	48	1	.	.	.
13	$(A_3\times A_1)''$	8	1	.	.	.
14	A_4	12	14 (1)	.	1	.
15	D_4	48	4, 1 (1)	15 (1)	.	.
16	$A_2\times A_1\times A_1\times A_1$	12	4 (1)	5	.	.
17	$A_2\times A_2\times A_1$	4	11 (1)	2	.	.
18	$A_3\times A_1\times A_1$	4	1	.	.	.
19	$A_3\times A_2$	4	4 (1)	8	.	.
20	$A_4\times A_1$	2	14 (1)	.	2	.
21	$D_4\times A_1$	8	4, 1 (1)	.	.	.
22	A_5'	12	11 (1)	5	.	.
23	A_5''	4	11 (1)	6	.	.
24	D_5	4	1, 4	.	.	.
25	$A_3\times A_2\times A_1$	2	4 (1)	12	.	.
26	$A_4\times A_2$	2	26 (1)	14	4	.
27	$A_5\times A_1$	2	11 (1)	9	.	.
28	$D_5\times A_1$	2	1, 4	.	.	.
29	A_6	2	29 (1)	.	.	1
30	D_6	2	14, 1, 4, 11 (1)	.	.	.
31	E_6	2	31, 11 (1)	15, 1 (1)	.	.
32	E_7	1	31, 1, 4, 11, 14, 26, 29 (1)	32, 5 (5)	.	.

Table 2: Decomposition matrix for E_7.

		β_{KK}	$p=2$	$p=3$	$p=5$	$p=7$
1	1	696729600	.	.	.	.
2	A_1	2903040	1	.	.	.
3	$A_1\times A_1$	46080	1	.	.	.
4	A_2	103680	4 (1)	1	.	.
5	$A_1\times A_1\times A_1$	2304	1	.	.	.
6	$A_2\times A_1$	1440	4 (1)	2	.	.
7	A_3	3840	1	.	.	.
8	$A_1\times A_1\times A_1\times A_1$	384	1	.	.	.
9	$A_2\times A_1\times A_1$	96	4 (1)	3	.	.
10	$A_2\times A_2$	288	10 (1)	1	.	.
11	$A_3\times A_1$	96	1	.	.	.
12	A_4	240	12 (1)	.	1	.
13	D_4	1152	4, 1 (1)	13 (1)	.	.
14	$A_2\times A_1\times A_1\times A_1$	24	4 (1)	5	.	.
15	$A_2\times A_2\times A_1$	24	10 (1)	2	.	.
16	$A_3\times A_1\times A_1$	16	1	.	.	.
17	$A_3\times A_2$	16	4 (1)	7	.	.
18	$A_4\times A_1$	12	12 (1)	.	2	.
19	$D_4\times A_1$	48	4, 1 (1)	19 (2)	.	.
20	A_5	24	10 (1)	5	.	.
21	D_5	48	1, 4	.	.	.
22	$A_2\times A_2\times A_1\times A_1$	8	10 (1)	3	.	.
23	$A_3\times A_2\times A_1$	4	4 (1)	11	.	.
24	$A_4\times A_1\times A_1$	4	12 (1)	.	3	.
25	$A_3\times A_3$	8	1	.	.	.
26	$A_4\times A_2$	4	26 (1)	12	4	.
27	$D_4\times A_2$	12	10, 4 (1)	13 (1)	.	.
28	$A_5\times A_1$	4	10 (1)	8	.	.
29	$D_5\times A_1$	4	1, 4	.	.	.
30	A_6	4	30 (1)	.	.	1
31	D_6	8	12, 1, 4, 10 (1)	.	.	.
32	E_6	12	32, 10 (1)	13, 1 (1)	.	.
33	$A_4\times A_2\times A_1$	2	26 (1)	18	6	.
34	$A_4\times A_3$	2	12 (1)	.	7	.
35	$A_6\times A_1$	2	30 (1)	.	.	2
36	$D_5\times A_2$	2	4, 10 (1)	21	.	.
37	A_7	2	1	.	.	.
38	$E_6\times A_1$	2	32, 10 (1)	19, 2 (2)	.	.
39	D_7	2	10, 1, 4, 12 (1)	.	.	.
40	E_7	2	32, 1, 4, 10, 12, 26, 30 (1)	40, 5 (5)	.	.
41	E_8	1	41, 1, 4, 10, 12, 26, 30, 32 (1)	41, 1, 13 (1)	41, 1 (1)	.

Table 3: Decomposition matrix for E_8.

		β_{KK}	$p=2$	$p=3$
1	1	1152	.	.
2	A_1'	48	1	.
3	A_1''	48	1	.
4	$A_1 \times A_1$	4	1	.
5	A_2'	12	5 (1)	1
6	A_2''	12	6 (1)	1
7	B_2	8	1	.
8	$(A_2 \times A_1)'$	2	5 (1)	2
9	$(A_2 \times A_1)''$	2	6 (1)	3
10	B_3'	2	5, 1 (1)	.
11	B_3''	2	6, 1 (1)	.
12	F_4	1	12, 1, 5, 6 (1)	12, 1 (1)

Table 4: Decomposition matrix for F_4.

		β_{KK}	$p=2$	$p=3$	$p=5$
1	1	120	.	.	.
2	A_1	4	1	.	.
3	$A_1 \times A_1$	2	1	.	.
4	A_2	2	4 (1)	1	.
5	$I_2(5)$	2	5 (1)	.	1
6	H_3	1	5, 1, 4 (1)	.	.

Table 5: Decomposition matrix for H_3.

		β_{KK}	$p=2$	$p=3$	$p=5$
1	1	14400	.	.	.
2	A_1	120	1	.	.
3	$A_1 \times A_1$	8	1	.	.
4	A_2	12	4 (1)	1	.
5	$I_2(5)$	20	5 (1)	.	1
6	$A_2 \times A_1$	2	4 (1)	2	.
7	$I_2(5) \times A_1$	2	5 (1)	.	2
8	A_3	2	1	.	.
9	H_3	2	5, 1, 4 (1)	.	.
10	H_4	1	10, 1, 4, 5 (1)	10, 1 (1)	10, 1 (1)

Table 6: Decomposition matrix for H_4.

zero in any other cases; such a calculation awaits a more detailed study of these algebras.

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