DOUBLY TRANSITIVE BUT NOT DOUBLY PRIMITIVE PERMUTATION GROUPS

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The connection between doubly transitive permutation groups G on a finite set Ω which are not doubly primitive and automorphism groups of block designs in which $\lambda = 1$ has been investigated by Sims [**2**] and Atkinson [**1**]. If, for $\alpha \in \Omega$, G_α has a set of imprimitivity of size 2 then it is easy to show that G is either sharply doubly transitive or is a group of automorphisms of a non-trivial block design on Ω in which $\lambda = 1$. In [**1**], in the proof of Theorem B, a simple argument due to G. Higman was used to establish the same conclusion if G_α has a set of imprimitivity of size 3. We shall continue the same investigation by proving the following theorem.

THEOREM. *Let G be a doubly transitive permutation group of degree n on a set Ω. Suppose that, for $\alpha \in \Omega$, G_α has a set of imprimitivity, Δ, of size 4; then G is either sharply doubly transitive or a group of automorphisms of a non-trivial block design on Ω in which $\lambda = 1$.*

We begin with a lemma which applies to any doubly transitive group.

LEMMA. *If θ and ϕ are distinct points of Ω, $G_{\{\theta, \phi\}}$ has at most one orbit of odd length on Ω.*

Proof. Suppose that Γ_1 and Γ_2 are distinct orbits of $G_{\{\theta, \phi\}}$ of odd length. If $\mu \in \Gamma_1$ then $G_{\{\theta, \phi\}\,\mu}$ has odd index in $G_{\{\theta, \phi\}}$. Under the action of $G_{\{\theta, \phi\}\,\mu}$ Γ_2 is a union of orbits not all of which have even length and so there is a point $\nu \in \Gamma_2$ such that $G_{\{\theta, \phi\}\mu\nu}$ has odd index in $G_{\{\theta, \phi\}\mu}$. Thus $G_{\{\theta, \phi\}\mu\nu}$ has index $\frac{1}{2}n(n-1)\,u$ in G, where u is odd. But then $G_{\{\theta, \phi\}\mu\nu}$ has index $\frac{1}{2}u$ in $G_{\mu\nu}$ which is evidently absurd.

Of course, the same argument shows that if Γ is an orbit of odd length of $G_{\{\theta, \phi\}}$ then, if $\gamma \in \Gamma$, all the orbits of $G_{\{\theta, \phi\}\gamma}$ on Ω have even length. The lemma is similar to Lemma 4 of Wagner [**3**] and, as in that paper, can be extended to the following: if G is k-fold transitive on Ω, H is the subgroup which preserves the k-element subset Σ of Ω and p is any prime not exceeding k, then H has at most $p-1$ orbits of size coprime to p on $\Omega - \Sigma$.

Proof of Theorem. Let $\Sigma_1 = \{(\alpha g,\ \Delta g) \mid g \in G\}$ and $\Sigma_2 = \{\Delta g \mid g \in G\}$. Then $|\Sigma_1| = n(n-1)/4$ and, if t is the number of elements of Σ_1 with a fixed second component, $|\Sigma_2| = n(n-1)/(4t)$. In the block design on Ω whose block set is Σ_2 the incidence equations give that $\lambda = 3/t$. For the rest of the proof we may, therefore, assume that $t = 1$.

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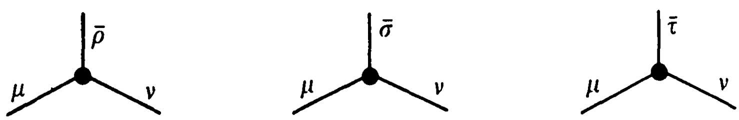
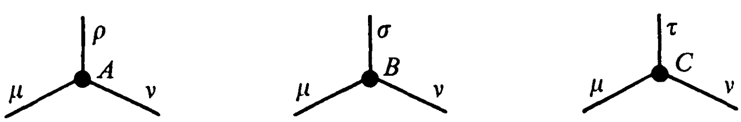
We construct a graph whose vertices are the 2-element subsets of Ω. Two vertices $\{\beta, \gamma\}$ and $\{\delta, \varepsilon\}$ are joined by an edge labelled αg if $\{\beta, \gamma, \delta, \varepsilon\} = \Delta g$. Then, since $t = 1$, each edge has a unique label and each vertex has valency 3. It is clear that G is a group of automorphisms of the labelled graph, transitive on vertices.

Let $\{\mu, \nu\}$ be any vertex and let ρ, σ, τ be the labels on the edges out of $\{\mu, \nu\}$. We may assume that $\{\rho, \sigma, \tau\}$ is an orbit of $G_{\{\mu, \nu\}}$; for $\{\rho, \sigma, \tau\}$ is certainly invariant under $G_{\{\mu, \nu\}}$ and if this group fixes one of these points we have that G is either sharply doubly transitive or the fixed point set of $G_{\mu\nu}$ together with its images under G form the blocks of a non-trivial design on Ω in which $\lambda = 1$.

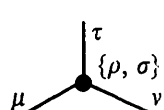
An obvious counting argument demonstrates that there are precisely 3 vertices where an edge labelled μ meets an edge labelled ν.



Clearly $\{\bar{\rho}, \bar{\sigma}, \bar{\tau}\}$ is invariant under $G_{\{\mu, \nu\}}$ and, as above, we may assume that it is an orbit of $G_{\{\mu, \nu\}}$. By the lemma, $\{\rho, \sigma, \tau\} = \{\bar{\rho}, \bar{\sigma}, \bar{\tau}\}$ and we have the configurations



It follows that the stabiliser of vertex A is transitive on $\{\rho, \mu, \nu\}$, and similarly for B and C. Hence an edge labelled ν is going out from $\{\rho, \mu\}$ and from $\{\sigma, \mu\}$. So one of the edges going out from $\{\rho, \sigma\}$ must be labelled ν. Similarly one of the edges from $\{\rho, \sigma\}$ must be labelled μ. Since edges μ, ν meet only with edges ρ, σ, τ, we must have the situation



Now $G_{\{\mu, \nu\}}$ has an orbit $\{\rho, \sigma, \tau\}$ and $G_{\{\rho, \sigma\}}$ has an orbit $\{\mu, \nu, \tau\}$. Therefore the set stabiliser G_Γ of $\Gamma = \{\rho, \sigma, \tau, \mu, \nu\}$ acts doubly transitively on Γ. In the block design whose blocks are the images under G of Γ we have

$$\lambda n(n-1)/(5.4) = b = [G : G_\Gamma] = [G : G_{\mu\nu}]/[G_\Gamma : G_{\mu\nu}] = n(n-1)/(5.4)$$

and hence $\lambda = 1$.

I thank the referee for noticing an error in my original proof and for suggesting the argument of the last two paragraphs.

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