DOUBLY TRANSITIVE BUT NOT DOUBLY PRIMITIVE PERMUTATION GROUPS II

M. D. ATKINSON

1. *Introduction*

Many of the known doubly transitive permutation groups permute sets with some geometric or combinatoric structure and these structures often turn out to be block designs. For example, PSU $(3, q)$ in its representation of degree q^3+1 preserves a block design in which $\lambda = 1$ and $k = q+1$. Another example is PSL (n, q), $n > 2$, which preserves a block design in which $\lambda = 1$ and $k = q+1$. In these cases the blocks are the lines of the corresponding unitary and projective spaces. It is easy to see that no group of automorphisms of a block design in which $\lambda = 1$ can act doubly primitively on the points. On the other hand the Suzuki groups Sz (q) in their doubly transitive representations of degree q^2+1 provide examples of groups which are neither doubly primitive nor groups of automorphisms of block designs in which $\lambda = 1$. The same is true of certain soluble groups. These examples suggest the following

CONJECTURE. *A doubly transitive but not doubly primitive permutation group G on a set* Ω *has one of the following properties.*

1. *It is metacyclic of prime degree p and order* $p(p-1)$.

2. *For some prime p it is of degree* 2^p *and order* $2^p(2^p-1)$ *or* $2^p(2^p-1)\,p$.

3. *It is a group of automorphisms of a block design on* Ω *in which* $\lambda = 1$.

4. $\mathrm{Sz}(q) \leqslant G \leqslant \mathrm{Aut}(\mathrm{Sz}(q))$.

The conjecture is known to be true if the stabiliser of a point has a set of imprimitivity of size $d = 2$, 3 or 4 [**1** and **2**]. Although this supports the conjecture it does not provide very strong evidence because, for these values of d, the Suzuki groups do not arise. In Section 3 of this paper I verify the conjecture in the case $d = 8$, the first case in which a Suzuki group can arise. In Section 2 I consider the situation for a general value of d and obtain some partial results.

Throughout, G will denote a doubly transitive permutation group on a v-element set Ω. If $\alpha \in \Omega$, a set of imprimitivity for G_α will be called an α-block. The setwise stabiliser of a set Δ will be denoted by G_Δ. The term block design will always mean a non-trivial block design on Ω in which $\lambda = 1$.

2. *General Results*

A result of Sims [**9**] which we use often is the following

LEMMA 2.1. *If* $\Gamma \subseteq \Omega-\{\alpha, \beta\}$ *is an orbit of* $G_{\alpha\beta}$ *then one of the following statements is true.* 1. *G is a group of automorphisms of a block design.*

Received 25 October, 1973.

[J. LONDON MATH. SOC. (2), 10 (1975), 53–60]

2. *Every composition factor of $G_{\alpha\beta}$ is a composition factor of some subgroup of the group induced by $G_{\alpha\beta}$ in Γ.*

LEMMA 2.2. *If Δ is a proper subset of Ω of size greater than 2, the group induced in Δ is doubly transitive and, for α, $\beta \in \Delta$, $G_{\alpha\beta}$ preserves Δ, then the images of Δ under G form the blocks of a block design.*

Proof. If $k = |\Delta|$ and b is the number of distinct images of Δ, the images of Δ form a block design in our sense if $\lambda = bk(k-1)/(v(v-1)) = 1$. Since $b = [G : G_\Delta]$ we have

$$\begin{aligned}\lambda &= ([G : G_{\alpha\beta}]/[G_\Delta : G_{\alpha\beta}]) \,.\, k(k-1)/(v(v-1)) \\ &= \frac{v(v-1)}{k(k-1)} \cdot \frac{k(k-1)}{v(v-1)} \\ &= 1.\end{aligned}$$

LEMMA 2.3. *If G has a regular normal subgroup of odd order then either G is a metacyclic group of order $p(p-1)$ and prime degree p or G is a group of automorphisms of a block design.*

Proof. If $\alpha \in \Omega$ then $G = G_\alpha N$, N may be identified with Ω and G may be considered to be a group acting on the elementary Abelian p-group N in which the elements of N act by multiplication and the elements of G_α act by conjugation. If $|N| = p$ then $G/N \cong \text{Aut}(N)$ is cyclic of order $p-1$ whereas, if $|N| > p$, the cyclic subgroups of N and their cosets which lie in N form the blocks of a block design which is preserved by G.

LEMMA 2.4. *The conjecture is true for soluble groups.*

Proof. The soluble doubly transitive groups have been classified by Huppert [**6**]. In view of his result and Lemma 2.3 we may confine our attention to groups G which consist of semi-linear transformations of a field F of 2^n elements, where n is not prime. Let L be a subfield of order 2^m, $1 < m < n$. The group induced on L is doubly transitive since it contains the transformations $\alpha \to a\alpha + b$, $a, b \in L$, $a \neq 0$; moreover, G_{01} preserves L. By Lemma 2.2, the images of L under G form the blocks of a block design.

LEMMA 2.5. *If the smallest orbit, Γ, of $G_{\alpha\beta}$ in $\Omega - \{\alpha, \beta\}$ is of length p, for some prime p, and the group induced by $G_{\alpha\beta}$ in Γ is regular then G has one of the following properties*:

1. $G_{\alpha\beta\gamma} = 1$ *for all* $\gamma \in \Omega - \{\alpha, \beta\}$ (and such groups have been classified in [**5**, **7** and **10**]);

2. *G is a group of automorphisms of a block design.*

Proof. By Lemma 2.2 we may assume that $G_{\alpha\beta}$ is a p-group. If $\gamma \in \Gamma$ then $G_{\alpha\beta\gamma}$ is normal in $G_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$ is maximal among p-groups which fix three or more points. By an argument of Livingstone and Wagner [**8**; pp. 400–401, the proof of Theorem 3, case II] $N(G_{\alpha\beta\gamma})$ acts doubly transitively on the fixed point set, Δ, of $G_{\alpha\beta\gamma}$; moreover $G_{\alpha\beta}$ preserves Δ since $G_{\alpha\beta} \leqslant N(G_{\alpha\beta\gamma})$. By Lemma 2.2 we may assume that

$G_{\alpha\beta\gamma} = 1$. Therefore $|G_{\alpha\beta}| = |\Gamma|$ and so, by the minimality of Γ, $G_{\alpha\beta\gamma} = 1$ for all $\gamma \in \Omega - \{\alpha, \beta\}$.

In the remainder of this section we study a fixed system of imprimitivity for G_α in which the α-blocks have size d. A term such as "β-block" will now always refer to a set of imprimitivity for G_β which is an image of one of the members of the given system for G_α. If Γ is any set then $\Gamma^{\{2\}}$ denotes the set of unordered pairs from Γ. If X is any transitive permutation group and Γ is any union of orbits of some point stabiliser then Γ^* denotes the union of the orbits paired with those in Γ.

The following result is proved in [**1**].

LEMMA 2.6. *If $\alpha, \beta \in \Omega$ then $\Gamma\{\alpha, \beta\} = \{\gamma | \{\alpha, \beta\} \subseteq \gamma$-block$\}$ is a $G_{\{\alpha,\beta\}}$-invariant set of size $d-1$. If, moreover, the group induced by G_α in an α-block is doubly transitive, then $\Gamma\{\alpha, \beta\}$ is an orbit of both $G_{\{\alpha,\beta\}}$ and $G_{\alpha\beta}$.*

Now G acts transitively on $\Omega^{\{2\}}$ and, continuing with the notation of Lemma 2.6, $G_{\{\alpha,\beta\}}$ preserves $\Gamma\{\alpha, \beta\}^{\{2\}} \subseteq \Omega^{\{2\}}$. Thus $\Theta = \Gamma\{\alpha, \beta\}^{\{2\}}$ is a union of orbits of $G_{\{\alpha,\beta\}}$. In the next lemma we consider Θ^* and introduce some notation which we use for the remainder of this section.

LEMMA 2.7. (i) $\Theta^* = \{\{\gamma, \delta\} | \{\gamma, \delta\} \subseteq \alpha$*-block and* $\{\gamma, \delta\} \subseteq \beta$*-block*$\}$.

(ii) *If $I_1, \ldots, I_k$ are the intersections of α-blocks and β-blocks which have size greater than 1 and $d_i = |I_i|$ then*

$$\sum d_i(d_i - 1) = (d-1)(d-2).$$

(iii) *There is a length-preserving correspondence between the orbits of $G_{\{\alpha,\beta\}}$ on Θ and the orbits of $G_{\{\alpha,\beta\}}$ on $I_1^{\{2\}} \cup \ldots \cup I_k^{\{2\}}$.*

Proof. $\{\gamma, \delta\} \in \Theta^*$ if and only if $\{\alpha, \beta\} \in \Gamma\{\gamma, \delta\}^{\{2\}}$ and this occurs if and only if $\{\gamma, \delta\}$ is contained in an α-block and in a β-block. This proves (i); (ii) and (iii) follow since $\Theta^* = I_1^{\{2\}} \cup \ldots \cup I_k^{\{2\}}$ and the union is disjoint.

One consequence of the equation $\sum d_i(d_i - 1) = (d-1)(d-2)$ is that the α-blocks are distinct from the β-blocks. Mr. A. G. Williamson has pointed out that this gives a simple proof of a theorem of Marggraaf, that a primitive group G of degree n which contains an m-cycle g is at least $(n-m+1)$-fold transitive. Since a proof of this theorem is not easily accessible in the literature I give his proof below.

By a theorem of Jordan [**11**; p. 32] G is doubly transitive if $n > m$, and it is evident, arguing by induction, that the theorem will follow if G can be shown to be doubly primitive when $n > m+1$. This will follow from another theorem of Marggraaf [**11**; p. 34] if $m \leqslant n/2$ and so we now assume that $m > n/2$.

Let α be a point fixed by g and let Δ be an α-block of size k, $1 < k < n-1$, which contains a point γ not fixed by g. Then Δ contains no points which are fixed by g; for if δ were such a point then $g \in G_{\alpha\delta}$ would preserve Δ and Δ would contain γg^i for all i which, since $m > n/2$, is impossible. Hence Δ is a block for the action of $\langle g \rangle$ on the points moved by g and so consists of the points $\gamma g^{im/k}$, $i = 1, 2, \ldots, k$. If β is another point fixed by g and Δ' is a β-block of size k containing γ the same argument shows that $\Delta = \Delta'$ and this contradicts the distinctness of α-blocks and β-blocks.

Returning to the general situation, the maximum possible size of intersection of an α-block and β-block is $d-1$ and this case is considered in the following lemma.

LEMMA 2.8. *If $k = 1$ then either G is sharply doubly transitive or G is a group of automorphisms of a block design.*

Proof. There is a unique α-block Δ_1 and a unique β-block Δ_2 such that $|\Delta_1 \cap \Delta_2| = d-1$. This unique intersection is preserved by $G_{\{\alpha,\beta\}}$. Hence $G_{\alpha\beta}$ preserves Δ_1 and Δ_2 and thus fixes the unique point γ of $\Delta_1 - \Delta_1 \cap \Delta_2$ and the unique point δ of $\Delta_2 - \Delta_1 \cap \Delta_2$. By Lemma 2.1 we may assume that $G_{\alpha\beta}$ has no fixed points other than α and β and so $\gamma = \beta$, $\delta = \alpha$. Thus Δ_1 is the α-block which contains β and $\Delta_1 - \{\beta\} = \Delta_1 \cap \Delta_2$ is $G_{\{\alpha,\beta\}}$-invariant. Hence, by Lemma 2 of [1], G is a group of automorphisms of a block design.

In the next two lemmas we investigate situations where the equation $\sum d_i(d_i-1) = (d-1)(d-2)$ reduces to something simpler.

LEMMA 2.9. *If the permutation group induced in an α-block by G_α is triply transitive then $d_1 = d_2 = \ldots = d_k$.*

Proof. We use a result of Cameron [4] : if X is a transitive permutation group with a suborbit $\Phi(\alpha)$ on which X_α acts doubly transitively then X_α also acts doubly transitively on the paired suborbit $\Phi^*(\alpha)$. We apply this result with $X = G_\beta$ acting transitively on $\Omega - \{\beta\}$. Let $\Phi(\mu)$ be the subset consisting of those points not equal to β which belong to the μ-block containing β. This is a suborbit of X because $X_\mu = G_{\beta\mu}$ and, moreover, X_μ acts doubly transitively on it. Now

$$\gamma \in \Phi^*(\alpha) \Leftrightarrow \alpha \in \Phi(\gamma) \Leftrightarrow \alpha \text{ belongs to the } \gamma\text{-block containing } \beta \Leftrightarrow \gamma \in \Gamma\{\alpha, \beta\}.$$

Thus $\Phi^*(\alpha) = \Gamma\{\alpha, \beta\}$ and, by Cameron's result, $X_\alpha = G_{\alpha\beta}$ acts doubly transitively on $\Gamma\{\alpha, \beta\}$. Hence $G_{\{\alpha,\beta\}}$ is transitive on $\Gamma\{\alpha, \beta\}^{\{2\}}$ and so transitive on $I_1^{\{2\}} \cup \ldots \cup I_k^{\{2\}}$ by Lemma 2.7. But the $I_j^{\{2\}}$ are obviously sets of imprimitivity for the action of $G_{\{\alpha,\beta\}}$ and so all have the same size. Consequently all the I_j have the same size.

LEMMA 2.10. *If the permutation group induced in an α-block by G_α is alternating or symmetric then the common value of the d_i is 2, 3 or $d-1$. Furthermore, if either $\frac{1}{2}(d-1)(d-2) > v-2$ or $d \equiv 0(4)$ then G is a group of automorphisms of a block design.*

Proof. Clearly $G_{\alpha\beta}$ acts as the alternating or symmetric group of degree $d-1$ on $\Psi - \{\beta\}$ where Ψ is the α-block containing β. By another result of Cameron [4], similar to the one above with the property " double transitivity " replaced by " contains the alternating group ", $G_{\alpha\beta}$ acts as A_{d-1} or S_{d-1} on $\Gamma\{\alpha, \beta\}$. In proving the first part we may certainly assume that $d > 5$ and hence, by Lemma 2.9, $kd_1(d_1-1) = (d-1)(d-2)$, where d_1 is the size of each intersection of α-block and β-block of size greater than 1. The k intersections are permuted transitively by $G_{\{\alpha,\beta\}}$ and so, if I is any one of them, $[G_{\{\alpha,\beta\}} : G_{\{\alpha,\beta\}I}] = k$ and hence $[G_{\alpha\beta} : G_{\alpha\beta I}] \leqslant k$.

Let Δ be the α-block containing I and $H = \{x \in G_\alpha | \Delta x = \Delta\}$; then $[G_\alpha : H] = (v-1)/d$. Since $[G_\alpha : G_{\alpha\beta I}] \leqslant (v-1)k$ and $G_{\alpha\beta I} \leqslant H$ it follows that

$$[H : G_{\alpha\beta I}] \leqslant dk = d(d-1)(d-2)/(d_1(d_1-1)).$$

Now H acts on Δ as A_d or S_d and, for each of these permutation groups, the setwise stabiliser of a set of size d_1 has index $\binom{d}{d_1}$. Since $G_{\alpha\beta I}$ is contained in some such stabiliser we have

$$d(d-1)(d-2)/(d_1(d_1-1)) \geqslant [H : G_{\alpha\beta I}] \geqslant \binom{d}{d_1}.$$

It follows that $d-2 \geqslant \binom{d-2}{d_1-2}$ and hence $d_1 = 2, 3$ or $d-1$.

If $d_1 = 2$ or $d_1 = 3$ there are, respectively, $(d-1)(d-2)$ or $\frac{1}{2}(d-1)(d-2)$ points not equal to α or β which lie in intersections of α-blocks and β-blocks of size greater than 1. Hence $\frac{1}{2}(d-1)(d-2) > v-2$ implies that $d_1 = d-1$ and, by Lemma 2.8, G is then a group of automorphisms of a block design.

Now we assume that $d \equiv 0(4)$. Since both $d-1$ and $\frac{1}{2}(d-1)(d-2)$ are odd, a Sylow 2-subgroup T of $G_{\{\alpha,\beta\}}$ must fix a point γ of $\Gamma\{\alpha, \beta\}$ and one of the pairs $\{\mu, \nu\}$ of $\Gamma\{\alpha, \beta\}^{(2)*}$. Then T preserves $\Gamma\{\mu, \nu\}$ and so fixes a point γ_1 in the set. However, T cannot have more than one fixed point (this fact is frequently used in the next section) or it would be a 2-subgroup of a two point stabiliser which has smaller 2-component than $G_{\{\alpha,\beta\}}$; hence $\gamma = \gamma_1$. Now $\alpha, \beta, \gamma \in \Gamma\{\mu, \nu\}$ and $G_{\{\mu,\nu\}}$ acts as the alternating or symmetric group on $\Gamma\{\mu, \nu\}$; so there is a permutation in G which cyclically permutes α, β, γ. Since $\gamma \in \Gamma\{\alpha, \beta\}$ we have $\alpha \in \Gamma\{\beta, \gamma\}$. Also, since $G_{\alpha\beta}$ is transitive on $\Gamma\{\alpha, \beta\}$ we have $\alpha \in \Gamma\{\beta, \theta\}$ for all $\theta \in \Gamma\{\alpha, \beta\}$. Therefore, the α-block containing β consists of β together with $\Gamma\{\alpha, \beta\}$ and the conclusion follows from Lemma 2 of **[1]**.

Finally in this section, we consider a particular case in which the equation $\sum d_i(d_i-1) = (d-1)(d-2)$ can be solved explicitly.

LEMMA 2.11. *If $v = 2d+1$ then the 4 intersections of α-blocks and β-blocks have sizes $\frac{1}{2}(d-2), \frac{1}{2}d, \frac{1}{2}d, \frac{1}{2}d$ or $\frac{1}{2}(d-1), \frac{1}{2}(d-1), \frac{1}{2}(d-1), \frac{1}{2}(d+1)$ according as d is even or odd.*

Proof. Let $\Delta_1(\alpha)$, $\Delta_2(\alpha)$ be the 2 α-blocks and $\Delta_1(\beta)$, $\Delta_2(\beta)$ the 2 β-blocks, the notation being chosen so that $\alpha \in \Delta_1(\beta)$ and $\beta \in \Delta_1(\alpha)$. Suppose that $|\Delta_1(\alpha) \cap \Delta_1(\beta)| = a$, $|\Delta_2(\alpha) \cap \Delta_2(\beta)| = b$, $|\Delta_1(\alpha) \cap \Delta_2(\beta)| = c$, $|\Delta_2(\alpha) \cap \Delta_1(\beta)| = c$, the last two being equal because an element interchanging α and β interchanges $\Delta_1(\alpha)$ and $\Delta_1(\beta)$ and interchanges $\Delta_2(\alpha)$ and $\Delta_2(\beta)$. Then

$$a+c = d-1$$
$$b+c = d$$
$$a(a-1)+b(b-1)+2c(c-1) = (d-1)(d-2).$$

Eliminating b and c gives $(2a-d+1)(2a-d+2) = 0$, from which the result follows.

In **[1]** I considered insoluble doubly transitive groups of degree $p = 4q+1$, where p and q were prime, and showed that such a group, if not doubly primitive, was isomorphic to PSL (3, 3). The arguments almost proved that a doubly transitive but not doubly primitive group of degree $4q+1$, where q is prime, is either sharply doubly transitive or a group of automorphisms of a block design. I think it is worth recording that Lemma 2.11 in conjunction with the arguments of **[1]** and a small amount of calculation gives a complete proof of this result.

3. *The case* $d = 8$

THEOREM. *If G is a doubly transitive permutation group in which G_α has a set of imprimitivity of size* 8 *then one of the following statements is true.*

(1) *G is sharply doubly transitive.*

(2) *G is a group of automorphisms of a block design.*

(3) $G \cong \mathrm{Sz}(8)$ *or* $G \cong \mathrm{Aut}(\mathrm{Sz}(8))$.

We shall continue to use the previous notation with $d = 8$ and, in addition, assume that G is a group satisfying the conditions but not any of the conclusions of the theorem. By the results mentioned in the first section, G_α does not have a set of imprimitivity of size 2 or 4. Hence the group induced in an α-block by G_α is primitive, therefore doubly transitive, and so, by Lemma 2.6, $G_{\alpha\beta}$ acts transitively on $\Gamma\{\alpha, \beta\}$. Some of the possibilities for the action of $G_{\{\alpha, \beta\}}$ on $\Gamma\{\alpha, \beta\}$ are easy to rule out. The cyclic group of order 7 and the Frobenius group of order 21 are excluded because a Sylow 2-subgroup of $G_{\{\alpha, \beta\}}$ would then have more than one fixed point in $\Gamma\{\alpha, \beta\}$. Lemma 2.10 excludes A_7 and S_7; indeed, the argument of Lemma 2.10 excludes PSL (3, 2) and shows also that if $\theta \in \Gamma\{\alpha, \beta\}$ then $\alpha \notin \Gamma\{\beta, \theta\}$. Accordingly, we have the following result.

LEMMA 3.1. *$G_{\{\alpha, \beta\}}$ acts on $\Gamma\{\alpha, \beta\}$ as a Frobenius group of order* 14 or 42. *Moreover, for any $\theta \in \Gamma\{\alpha, \beta\}$, $\alpha \notin \Gamma\{\beta, \theta\}$.*

LEMMA 3.2. *Let $\{\mu, \nu\}$ be any unordered pair from Ω and let σ, τ be distinct elements of $\Gamma\{\mu, \nu\}$. Then $\Gamma\{\mu, \nu\}$ contains exactly one point of $\Gamma\{\sigma, \tau\}$. Moreover, a point π of $\Gamma\{\mu, \nu\}$ belongs to $\Gamma\{\sigma, \tau\}$ if and only if the group induced in $\Gamma\{\mu, \nu\}$ by $G_{\{\mu, \nu\}}$ contains a permutation which fixes π and interchanges σ and τ.*

Proof. $G_{\{\mu, \nu\}}$ acts on $\Gamma\{\mu, \nu\}$ as a Frobenius group of order 14 or 42. In both of these cases there is a Sylow 2-subgroup U of $G_{\{\mu, \nu\}}$ which preserves $\{\sigma, \tau\}$. Then U fixes a point π of $\Gamma\{\mu, \nu\}$ and a point in $\Gamma\{\sigma, \tau\}$. However, since U cannot fix more than one point, $\pi \in \Gamma\{\sigma, \tau\}$.

Let $V < U$ be a Sylow 2-subgroup of $G_{\sigma\tau}$. Since $V \leqslant G_{\{\mu, \nu\}}$ and fixes $\sigma, \tau \in \Gamma\{\mu, \nu\}$, V fixes $\Gamma\{\mu, \nu\}$ pointwise. If $\Gamma\{\mu, \nu\}$ contained more than one point of $\Gamma\{\sigma, \tau\}$ then V would fix at least 2 points of $\Gamma\{\sigma, \tau\}$ and so would fix $\Gamma\{\sigma, \tau\}$ pointwise. Thus $G_{\sigma\tau}$ would act on $\Gamma\{\sigma, \tau\}$ as a group of odd order and, by Lemma 2.1, $|G_{\sigma\tau}|$ would be odd. Bender's theorem [3] would now determine the structure of G and a check through the possibilities reveals that G would not be a counterexample.

Any permutation of $U - V$ will fix π and interchange σ and τ. Conversely, let $a \in G_{\{\mu, \nu\}}$ and let a fix $\rho \in \Gamma\{\mu, \nu\}$ and interchange σ and τ. Then $\pi a \in \Gamma\{\mu, \nu\} \cap \Gamma\{\sigma, \tau\}$ and so $\pi a = \pi$. But a has at most one fixed point in $\Gamma\{\mu, \nu\}$ and so $\pi = \rho$.

LEMMA 3.3. *Let $\{\sigma, \tau\}$ be any unordered pair from Ω and let $\{\mu_i, \nu_i\}$, $i = 1, 2, 3$, be 3 unordered pairs such that $\{\sigma, \tau\} \subseteq \Gamma\{\mu_i, \nu_i\}$, $i = 1, 2, 3$. Suppose there exists a point π in each $\Gamma\{\mu_i, \nu_i\}$ and also in $\Gamma\{\sigma, \tau\}$. Then the three points $\rho_i \in \Gamma\{\sigma, \pi\} \cap \Gamma\{\mu_i, \nu_i\}$, $i = 1, 2, 3$, are equal.*

Proof. Let σ, τ, π, ρ_1, χ_1, χ_2, χ_3 be the points of $\Gamma\{\mu_1, \nu_1\}$ and let X be the permutation group (a Frobenius group of order 14 or 42) induced by $G_{\{\mu_1, \nu_1\}}$ in this set. Since $\pi \in \Gamma\{\sigma, \tau\}$, X contains a permutation which interchanges σ and τ and fixes π; we may take this to be $a = \pi(\sigma, \tau)(\rho_1, \chi_1)(\chi_2, \chi_3)$. Similarly, because $\rho_1 \in \Gamma\{\sigma, \pi\}$, X contains a permutation which interchanges σ and π and fixes ρ_1; we may take this to be $b = \rho_1(\sigma, \pi)(\chi_1, \chi_2)(\chi_3, \tau)$. Then $ababa = \tau(\pi, \chi_2)(\chi_1, \chi_3)(\rho_1, \sigma)$ and therefore $\tau \in \Gamma\{\rho_1, \sigma\}$. Thus $\rho_1 \in \Sigma$, the set of 7 points not equal to σ of the τ-block containing σ. Moreover, no other point ζ of $\Gamma\{\mu_1, \nu_1\}$ belongs to Σ because this would imply that $\tau \in \Gamma\{\zeta, \sigma\}$ and X would contain a permutation fixing τ and interchanging ζ and σ which it does not. Similarly, ρ_2 and ρ_3 are the unique points of $\Gamma\{\mu_2, \nu_2\}$ and $\Gamma\{\mu_3, \nu_3\}$ which belong to Σ.

Let Π be the set of 21 pairs $\{\xi, \eta\}$ for which $\sigma, \tau \in \Gamma\{\xi, \eta\}$. By Lemma 2.7, $G_{\{\sigma, \tau\}}$ has 3 orbits of size 7 or is transitive on the 21 pairs of Π; since $[G_{\{\sigma, \tau\}} : G_{\sigma\tau}] = 2$ the same is true of $G_{\sigma\tau}$. Define an equivalence relation on Π by $\{\xi, \eta\} \sim \{\xi', \eta'\}$ if and only if $\Gamma\{\xi, \eta\}$ and $\Gamma\{\xi', \eta'\}$ contain the same point of $\Gamma\{\sigma, \tau\}$. This is a $G_{\sigma\tau}$-invariant relation and $G_{\sigma\tau}$ is transitive on the set of 7 equivalence classes. Each equivalence class thus consists of 3 pairs and the group induced by $G_{\sigma\tau}$ in each class is either transitive or trivial. Since $\{\mu_i, \nu_i\}$, $i = 1, 2, 3$, constitute one such class, the group induced by $G_{\sigma\tau}$ on these 3 pairs is either transitive or trivial.

Let K be the kernel of the action of $G_{\sigma\tau}$ on $\Gamma\{\sigma, \tau\}$. Since K fixes π, K permutes the 3 pairs $\{\mu_i, \nu_i\}$, $i = 1, 2, 3$. If K acts transitively on them then it must also act transitively on $\{\rho_1, \rho_2, \rho_3\}$. But a normal subgroup of $G_{\sigma\tau}$ can only have orbits of length 1 or 7 on Σ and so $\rho_1 = \rho_2 = \rho_3$. Consequently we may assume that K fixes each pair $\{\mu_i, \nu_i\}$.

Let T_1 be a Sylow 2-subgroup of $G_{\sigma\tau}$ which fixes $\{\mu_1, \nu_1\}$; then T_1 fixes $\Gamma\{\mu_1, \nu_1\}$ pointwise. We now show that T_1 also fixes each of $\{\mu_2, \nu_2\}$ and $\{\mu_3, \nu_3\}$. Let T_2 be a Sylow 2-subgroup of $G_{\sigma\tau}$ which fixes $\{\mu_2, \nu_2\}$. Then T_2 fixes $\Gamma\{\mu_2, \nu_2\}$ pointwise and, because $G_{\sigma\tau}$ as a group on $\Gamma\{\sigma, \tau\}$ has a unique Sylow 2-subgroup fixing π, we have $T_1 K = T_2 K$. By Sylow's theorem $T_1 = T_2^k$, $k \in K$, and since k fixes $\{\mu_2, \nu_2\}$ so also does T_1. Similarly T_1 fixes $\{\mu_3, \nu_3\}$.

Thus T_1 fixes $\{\rho_1, \rho_2, \rho_3\}$ pointwise. If ρ_1, ρ_2, ρ_3 are not all equal then, since T_1 is a Sylow 2-subgroup of $G_{\sigma\tau}$ and $\rho_1, \rho_2, \rho_3 \in \Gamma\{\sigma, \pi\}$, $G_{\sigma\pi}$ acts on $\Gamma\{\sigma, \pi\}$ as a group of odd order. The same contradiction as was reached at the end of the proof of Lemma 3.2 can now be obtained.

The proof of the theorem can now be completed. If $\theta \in \Gamma\{\alpha, \beta\}$ then, as a consequence of Lemma 3.2, there are precisely 3 pairs $\{\gamma_i, \delta_i\}$, $i = 1, 2, 3$, such that $\alpha, \beta, \theta \in \Gamma\{\gamma_i, \delta_i\}$, $i = 1, 2, 3$. Let α, β, θ, ϕ, ψ, ζ, η be the points of $\Gamma\{\gamma_1, \delta_1\}$. By Lemma 3.2 and an appropriate choice of notation we may assume that $\phi \in \Gamma\{\alpha, \theta\}$ and that the group induced by $G_{\{\gamma_1, \delta_1\}}$ in $\Gamma\{\gamma_1, \delta_1\}$ contains permutations

$$a = \theta(\alpha, \beta)(\phi, \psi)(\zeta, \eta) \qquad \text{and} \qquad b = \phi(\alpha, \theta)(\beta, \zeta)(\psi, \eta).$$

Then, again by Lemma 3.2, consideration of a, aba, $babab$, $abababa$ shows that $\theta \in \Gamma\{\phi, \psi\}, \psi \in \Gamma\{\beta, \theta\}, \eta \in \Gamma\{\theta, \psi\}, \zeta \in \Gamma\{\theta, \phi\}$.

We now apply Lemma 3.3 four times each with $\{\gamma_i, \delta_i\}$ in the role of $\{\mu_i, \nu_i\}$, $i = 1, 2, 3$. Firstly, with α, β, $\theta = \sigma$, τ, π respectively, $\phi \in \Gamma\{\alpha, \theta\}$ implies that

$\phi \in \Gamma\{\gamma_i, \delta_i\}$, $i = 2, 3$. Secondly, with $\beta, \alpha, \theta = \sigma, \tau, \pi$ respectively, $\psi \in \Gamma\{\beta, \theta\}$ implies that $\psi \in \Gamma\{\gamma_i, \delta_i\}$, $i = 2, 3$. Thirdly, with $\psi, \phi, \theta = \sigma, \tau, \pi$ respectively, $\eta \in \Gamma\{\theta, \phi\}$ implies that $\eta \in \Gamma\{\gamma_i, \delta_i\}$, $i = 2, 3$. Fourthly, with $\phi, \psi, \theta = \sigma, \tau, \pi$ respectively, $\zeta \in \Gamma\{\theta, \phi\}$ implies that $\zeta \in \Gamma\{\gamma_i, \delta_i\}$, $i = 2, 3$.

Hence $\Gamma\{\gamma_1, \delta_1\} = \Gamma\{\gamma_2, \delta_2\} = \Gamma\{\gamma_3, \delta_3\}$ and this set is equal to no other $\Gamma\{\chi, \chi'\}$. Therefore $G_{\{\gamma_1, \delta_1\}}$ has an invariant set $\{\gamma_2, \delta_2, \gamma_3, \delta_3\}$ and Lemma 2.1 gives the final contradiction.

References

1. M. D. Atkinson, " Two theorems on doubly transitive permutation groups ", *J. London Math Soc.*, 6 (1973), 269–274.
2. ———, " Doubly transitive but not doubly primitive permutation groups ", *J. London Math. Soc.*, 7 (1974), 632–634.
3. H. Bender, " Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt ", *J. Algebra*, 17 (1971), 527–554.
4. P. J. Cameron, " Permutation groups with multiply transitive suborbits ", *Proc. London Math. Soc.*, 25 (1972), 427–440.
5. W. Feit, " On a class of doubly transitive permutation groups ", *Illinois J. Math.*, 4 (1960), 170–186.
6. B. Huppert, " Zweifach transitive, auflösbare Permutationsgruppen ", *Math. Z.*, 68 (1957), 126–150.
7. N. Itô, " On a class of doubly transitive permutation groups ", *Illinois J. Math.*, 6 (1962), 341–352.
8. D. Livingstone and A. Wagner, " Transitivity of finite permutation groups on unordered sets ", *Math. Z.*, 90 (1965), 393–403.
9. C. C. Sims, " Computational methods in the study of permutation groups ", *Computational problems in abstract algebra*, 169–183, John Leech (Editor) (Pergamon Press, 1970).
10. M. Suzuki, " On a class of doubly transitive groups ", *Ann. of Math.*, 75 (1962), 105–145.
11. H. Wielandt, *Finite permutation groups* (New York, Academic Press, 1964).

Department of Computing Mathematics,
University College, Cardiff.