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ON RANK 3 GROUPS HAVING $\lambda = 0$

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In this paper we shall consider certain rank 3 permutation groups G which act on a set Ω of size n. Thus a point stabiliser G_α will have 3 orbits $\{\alpha\}$, $\Delta(\alpha)$, $\Gamma(\alpha)$ of sizes 1, k, l respectively. It is well known that, if $|G|$ is even, then the orbital Δ defines a strongly regular graph on Ω. In this graph, every point has valency k, every pair of adjacent points are adjacent to a constant number λ of common points, and every pair of non-adjacent points are adjacent to a constant number μ of common points. This notation is reasonably standard (see [**4**], where much background theory is given).

It is also well known and easy to prove that a primitive rank 3 group G in which $G_\alpha^{\Delta(\alpha)}$ is doubly transitive has $\lambda = 0$. Then the associated strongly regular graph has no triangles: such graphs are discussed in Chapter 4 of [**2**]. The known examples of primitive rank 3 groups of even order with $\lambda = 0$ are as follows:

TABLE 1

G	n	k
Dihedral of order 10	5	2
A_5, S_5	10	3
$2^4.F_{20}$, $2^4.A_5$, $2^4.S_5$	16	5
$PSU(3, 5)$, $PSU(3, 5).2$	50	7
$PSL(3, 4) \leqslant G \leqslant PSL(3, 4).V_4$	56	10
M_{22}, $M_{22}.2$	77	16
HS, $HS.2$	100	22

The notation in this table is standard except for F_{20}, the Frobenius group of degree 5 and order 20. In all these examples G_α acts doubly transitively on $\Delta(\alpha)$. In Theorem 1 a sufficient condition is given for $G_\alpha^{\Delta(\alpha)}$ to be doubly transitive. Then primitive rank 3 groups with $\lambda = 0$ of degree less than 1000 are investigated; it can be shown that if there are any such groups in addition to those of Table 1 then $G_\alpha^{\Delta(\alpha)}$ is a new doubly transitive group. The justification for this may be found in [**1**], but here we only consider two cases. In view of the results announced in [**5**] it seems unlikely that such groups exist.

LEMMA. *Let G be a rank 3 group of even order with $\lambda = 0$. For some prime p suppose that $p^t | l$, $p \nmid k$. Then the non-trivial subdegrees of $G_\alpha^{\Delta(\alpha)}$ are divisible by p^t. Moreover, if $p = 2$ and μ is even, these subdegrees are divisible by 2^{t+1}.*

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Proof. Let P be a Sylow p-subgroup of G_α. Since $p \nmid k$, P fixes a point β of $\Delta(\alpha)$ and is a Sylow p-subgroup of G_β. Let $\Sigma \subseteq \Delta(\alpha) - \{\beta\}$ be an orbit of $G_{\alpha\beta}$.

Assume first that $|\Sigma|$ is not divisible by p^t. Then, for some $\sigma \in \Sigma$, $Q = P_\sigma$ is a p-subgroup of index in G_β not divisible by p^t. However, as $\lambda = 0$, $\sigma \in \Gamma(\beta)$ and so Q is contained in a subgroup of index l in G_β. Since $p^t|l$ this is a contradiction. Hence $|\Sigma|$ is divisible by p^t.

Assume next that $p = 2$, μ is even and that 2^{t+1} does not divide $|\Sigma|$. Since $|\Sigma|$ is even, the number $\binom{|\Sigma|}{2}$ of unordered pairs from Σ is not divisible by 2^t. Hence there exist $\sigma, \tau \in \Sigma$ such that $R = P_{\{\sigma,\tau\}}$ has index in G_α not divisible by 2^t. Since $\lambda = 0$, σ and τ are not adjacent in the Δ-graph and R permutes the μ points joined to both. One of these points is α which is fixed by R and, as μ is even, R must fix another point γ. Since $\lambda = 0$, $\gamma \in \Gamma(\alpha)$ and so R is contained in a subgroup of G_α of index l; again this is a contradiction and so 2^{t+1} divides $|\Sigma|$.

As a consequence of this lemma and the relation $\mu l = k(k - 1)$ the following is true:

THEOREM 1. *If G is a rank 3 group of even order with $\lambda = 0$ and $\mu|2k$ then G_α acts doubly transitively on $\Delta(\alpha)$.*

The 15 parameter sets for primitive rank 3 groups of even order with $\lambda = 0$ and $100 < n \leqq 1000$ which satisfy all the criteria of [**4**] are as follows:

TABLE 2

n	k	l	μ	n	k	l	μ	n	k	l	μ
162	21	140	3	352	26	325	2	650	55	594	5
176	25	150	4	352	36	315	4	667	96	570	16
210	33	176	6	392	46	345	6	704	37	666	2
266	45	220	9	552	76	475	12	784	116	667	20
324	57	266	12	638	49	588	4	800	85	714	10

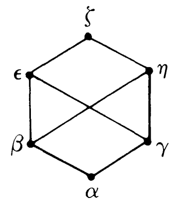
In addition to the parameter sets for the groups in Table 1 there are two further parameter sets with $n \leqq 100$ (having $n = 28$ and $n = 64$) but it is fairly well-known and easy to prove that the corresponding strongly regular graphs do not exist.

THEOREM 2. *If G is a rank 3 group with a parameter set belonging to Table 2 then G_α acts doubly transitively on $\Delta(\alpha)$ and, moreover, $G_\alpha^{\Delta(\alpha)}$ is not isomorphic to any known doubly transitive group.*

The proof of this result requires a detailed case by case study of each parameter set. We shall omit nearly all the details (for which, see [**1**]) and just give two examples to illustrate some of the methods.

Example 1. In the parameter set which has $n = 162$, $G_\alpha^{\Delta(\alpha)}$ is not one of the known doubly transitive groups.

We argue with the associated strongly regular graph. Suppose that β, γ are distinct points of $\Delta(\alpha)$. Then α and two further points ϵ, η are joined to β and γ. Since $\lambda = 0$ neither of ϵ, η are joined to α and they are not joined to each other. In addition to β and γ there is another point ζ joined to each of ϵ, η.



Clearly, $G_{\alpha\{\beta,\gamma\}}$ fixes ζ. Since $[G_\alpha : G_{\alpha\{\beta,\gamma\}}] = 21.20/2 = 210$ and 140 does not divide 210, ζ does not belong to $\Gamma(\alpha)$ i.e. $G_{\alpha\{\beta,\gamma\}}$ fixes a point of $\Delta(\alpha)$. But this property is not shared by any of the known doubly transitive groups of degree 21.

Example 2. In the parameter set which has $n = 784$, $G_\alpha^{\Delta(\alpha)}$ is doubly transitive.

Since $116 = 2^2 29$ and $667 = 23.29$ the lemma above shows that the non-trivial subdegrees of $G_\alpha^{\Delta(\alpha)}$ are divisible by 23 (and so $G_\alpha^{\Delta(\alpha)}$ must be primitive). If these subdegrees were 1, 46, 69 Higman's criteria would provide a contradiction. In all other cases there is a subdegree 23. By [**3**] and a theorem of Burnside this must correspond to a soluble constituent. Hence 2, 11, 23, 29 are the only primes which can divide the order of the insoluble group $G_\alpha^{\Delta(\alpha)}$; this contradicts Thompson's classification of N-groups.

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