TWO THEOREMS ON DOUBLY TRANSITIVE PERMUTATION GROUPS

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In a series of papers [**3**, **4** and **5**] on insoluble (transitive) permutation groups of degree $p = 2q+1$, where p and q are primes, N. Itô has shown that, apart from a small number of exceptions, such a group must be at least quadruply transitive. One of the results which he uses is that an insoluble group of degree $p = 2q+1$ which is not doubly primitive must be isomorphic to PSL (3, 2) with $p = 7$. This result is due to H. Wielandt, and Itô gives a proof in [**3**]. It is quite easy to extend this proof to give the following result: a doubly transitive group of degree $2q+1$, where q is prime, which is not doubly primitive, is either sharply doubly transitive or a group of automorphisms of a block design with $\lambda = 1$ and $k = 3$. Our notation for the parameters of a block design, v, b, k, r, λ, is standard; see [**9**].

In this paper we shall prove two results about doubly transitive but not doubly primitive groups which resemble the two results mentioned above.

THEOREM A. *Let G be an insoluble transitive permutation group of degree* $p = 4q+1$, *where p and q are primes, which is not doubly primitive. Then* $G \cong$ PSL (3, 3) *and* $p = 13$.

THEOREM B. *Let G be a doubly transitive permutation group on* Ω *of degree* $3q+1$, *where q is a prime. Then one of the following statements is true.*

(1) *G is doubly primitive.*

(2) *G is sharply doubly transitive.*

(3) *G is a group of automorphisms of a block design on* Ω *with* $\lambda = 1$ *and* $k = 4$.

(4) $G \cong$ PSL (3, 2) *and* $q = 2$.

Theorem B has the following consequence.

COROLLARY C. *Let G be an insoluble doubly transitive permutation group of degree* $3q+1$, *where q is a prime and* $q \equiv 3(4)$. *Then G is doubly primitive.*

To prove this we have to exclude all but the first alternative of Theorem B. The insolubility of G excludes possibility (2), possibility (3) does not occur because the incidence equations of the design imply $q \equiv 1(4)$ and possibility (4) obviously does not arise.

If G is a doubly primitive group of degree $3q+1$, where q is prime, then the stabiliser of a point acts as a primitive group of degree $3q$ on the remaining points. P. M. Neumann has considered primitive groups of degree $3q$, [**7**], and his results imply that, in many cases, G is triply transitive.

The proofs of Theorems A and B are mainly combinatorial; however, at the end of the proof of Theorem A I use the following unpublished theorem of P. M. Neumann which he has proved using modular character theory. I take this opportunity of thanking him for showing me the proof.

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THEOREM of P. M. Neumann. *Let G be an insoluble (transitive) permutation group on* Ω *of prime degree p. Let P be a Sylow p-subgroup of G and assume that* $|N(P)| = kp$, *where k is odd. Then, if* $\alpha, \beta \in \Omega$ *and* $\alpha \neq \beta$, $G_{\alpha\beta}$ *has at most* $(p-1)/k$ *orbits in* $\Omega - \{\alpha, \beta\}$.

By using this theorem earlier in the proof of Theorem A one could slightly shorten the proof. I have preferred not to do this not only because most of the proof then relies on elementary arguments, but also because the arguments are largely independent of the fact that p is prime and so subsequently could be incorporated into a proof of a more general result.

Throughout this paper the term " block " is used only in the block design sense; however, a term such as " K-block " refers to a set of imprimitivity for a group K. Before giving the proofs of Theorems A and B we prove a series of preliminary lemmas.

LEMMA 1 (E. Witt [12]). *Let X be a doubly transitive group on a set* Ω, *let* $\alpha, \beta \in \Omega$ *with* $\alpha \neq \beta$ *and let K be a weakly closed subgroup of* $X_{\alpha\beta}$. *Then, if* $\Delta = \text{fix}(K)$, *in the block design whose blocks are the images under X of* Δ *we have* $\lambda = 1$.

Proof. Suppose that $\{\alpha, \beta\} \subseteq \Delta x$ for some $x \in X$. Then α and β are fixed by K^x and, by the weak closure, $K^x = K$. Thus $\Delta = \Delta x$ and so only one block contains $\{\alpha, \beta\}$.

We note that $X_{\alpha\beta}$ itself and any normal Sylow p-subgroup of $X_{\alpha\beta}$ are weakly closed in $X_{\alpha\beta}$.

LEMMA 2. *Let X be a doubly transitive group on a set* Ω, *let* $\alpha \in \Omega$ *and let* Δ *be a set of imprimitivity for the action of* X_α *on* $\Omega - \{\alpha\}$. *Let* $\beta \in \Delta$ *and suppose that* $\Delta - \{\beta\}$ *is invariant under* $X_{\{\alpha, \beta\}}$. *Then, in the block design whose blocks are the images under X of* $\Gamma = \Delta \cup \{\alpha\}$ *we have* $\lambda = 1$.

Proof. Let $Y = \{x \in X \mid \Gamma x = \Gamma\}$. Because Δ is an X_α-block we have $X_{\alpha\beta} \leqslant Y$ and Y_α transitive on Δ. Furthermore, as Y contains an element which interchanges α and β, Y is doubly transitive on Γ. We have

$$\frac{\lambda v(v-1)}{k(k-1)} = b = [X : Y] = \frac{[X : X_{\alpha\beta}]}{[Y : X_{\alpha\beta}]} = \frac{v(v-1)}{k(k-1)}.$$

Thus $\lambda = 1$.

The next lemma is an extension of a result of C. C. Sims (Theorem 3.7 of [9]).

LEMMA 3. *Let X be a doubly transitive group on a set* Ω, *let* $\alpha \in \Omega$ *and let* Δ *be a set of imprimitivity of size m for the action of* X_α *on* $\Omega - \{\alpha\}$. *Then, if* $\beta \in \Delta$, $X_{\{\alpha, \beta\}}$ *has an invariant set* Γ *of size* $m-1$ *on* $\Omega - \{\alpha, \beta\}$. *Furthermore, if* $X_{\alpha\beta}$ *is transitive on* $\Delta - \{\beta\}$, $X_{\alpha\beta}$ *and* $X_{\{\alpha, \beta\}}$ *are transitive on* Γ.

Proof. A routine check establishes that, for any $x, y \in X$, Δx is an $X_{\alpha x}$-block and, if $\alpha x = \alpha y$, that Δx and Δy belong to the same block system of $X_{\alpha x}$. The set $B = \{(\alpha x, \Delta x) \mid x \in X\}$ obviously has size $n(n-1)/m$, where $n = |\Omega|$. We may take the elements of B as the blocks of a block design on Ω if we agree that a point $\omega \in \Omega$ is incident with the block $(\alpha x, \Delta x)$ if and only if $\omega \in \Delta x$. The incidence equations for the design tell us that $\lambda = m-1$. Thus there are exactly $m-1$ blocks

$(\alpha x_i, \Delta x_i)$, $i = 1, 2, \ldots, m-1$, with $\{\alpha, \beta\} \subseteq \Delta x_i$ and it is clear that all the αx_i, $i = 1, 2, \ldots, m-1$, are distinct and form an $X_{\{\alpha, \beta\}}$-invariant set Γ.

Now assume that $X_{\alpha\beta}$ is transitive on $\Delta - \{\beta\}$ and let $\gamma_1, \gamma_2 \in \Gamma$. By definition there exist elements $x_1, x_2 \in X$ such that $\alpha x_1 = \gamma_1$, $\alpha x_2 = \gamma_2$ and $\{\alpha, \beta\}$ is contained in both Δx_1 and Δx_2. Since X is doubly transitive there exists $a \in X_\alpha$ such that $\gamma_1 a = \gamma_2$. Now $(\gamma_1 a, \Delta x_1 a) = (\gamma_2, \Delta x_1 a)$ and $(\gamma_2, \Delta x_2)$ are both members of B and $\alpha \in \Delta x_2 \cap \Delta x_1 a$. Thus $\Delta x_2 = \Delta x_1 a$ and $\{\alpha, \beta, \beta a) \subseteq \Delta x_2 = \Delta x_1 a$. However, a consequence of our extra assumption is that $X_{\gamma_2 \alpha}$ is transitive on $\Delta x_2 - \{\alpha\}$. Therefore we can find $b \in X_{\gamma_2 \alpha}$ such that $\beta ab = \beta$. We then have $ab \in X_{\alpha\beta}$ and $\gamma_1 ab = \gamma_2$; thus $X_{\alpha\beta}$ is transitive on Γ as also is $X_{\{\alpha, \beta\}}$.

The proof of the next lemma follows directly from the incidence equations of a block design.

LEMMA 4. *Let Ω be a set on which there is a non-trivial block design with $\lambda = 1$. Then*

(a) *if $|\Omega| = p = 4q+1$, where p and q are prime, then $p = 13$,*

(b) *if $|\Omega| = 3q+1$, where q is prime, then $k = 4$ or $q = 2$.*

We shall frequently use the well-known theorem of Burnside that a transitive group of prime degree is either doubly transitive or is a metacyclic Frobenius group.

Proof of Theorem A

Let G be an insoluble transitive permutation group on a set Ω of size $p = 4q+1$, where p and q are prime, which is not doubly primitive. If G is a counterexample to Theorem A we may assume, from a search through the list of groups of degree 13 (see, for example, [9]), that $p \neq 13$. In fact, we may assume that $p > 53$ from the results of [1]. From Lemma 4 we may, in addition, assume that G is not a group of automorphisms of a non-trivial block design with $\lambda = 1$. Furthermore, by a theorem of [1] we have that q divides $|G|$ to the first power only.

Let P be a Sylow p-subgroup of G and Q a Sylow q-subgroup of G_α where $\alpha \in \Omega$. Let $\Delta_1, \Delta_2, \ldots$ be a non-trivial system of imprimitivity for the action of G_α on $\Omega - \{\alpha\}$. Let $H = \{x \in G_\alpha \mid \Delta_1 x = \Delta_1\}$, $K = \{x \in G_\alpha \mid \Delta_i x = \Delta_i, i = 1, 2, \ldots\}$ and let $\beta \in \Delta_1$. Then $G_{\alpha\beta} \leqslant H$ and $K \lhd G_\alpha$. There are four cases to consider depending on the size of the G_α-blocks.

Case 1. $2q$ G_α-blocks of size 2

If $\Delta_1 = \{\beta, \gamma\}$ then $G_{\alpha\beta}$ fixes γ and Lemma 1 provides a contradiction.

Case 2. q G_α-blocks of size 4

By case 1 we may assume that H acts primitively on Δ_1 and so acts as A_4 or S_4 on Δ_1. Therefore $\Delta_1 - \{\beta\}$ is an orbit of $G_{\alpha\beta}$.

If G_α is insoluble then it is also insoluble as a permutation group on the set $\{\Delta_1, \ldots, \Delta_q\}$ and so, in this case, it acts doubly transitively on this set. Consequently, H permutes $\{\Delta_2, \ldots, \Delta_q\}$ transitively and, as $[H : G_{\alpha\beta}] = 4$, all the orbits of $G_{\alpha\beta}$ on $\{\Delta_2, \ldots, \Delta_q\}$ have size at least $(q-1)/4 > 3$. It follows that $\Delta_1 - \{\beta\}$ is the unique orbit of $G_{\alpha\beta}$ of size 3 and therefore $\Delta_1 - \{\beta\}$ is $G_{\{\alpha, \beta\}}$-invariant. We may now obtain a contradiction from Lemma 2.

If G_α is soluble then H permutes the set $\{\Delta_2, \ldots, \Delta_q\}$ semi-regularly and therefore $G_{\alpha\beta}$ also permutes $\{\Delta_2, \ldots, \Delta_q\}$ semi-regularly. If $\Delta_1 - \{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant then we may obtain the same contradiction as above from applying Lemma 2. Hence we may assume that $G_{\alpha\beta}$ possesses another orbit Γ of size 3 such that $\Gamma \cup (\Delta_1 - \{\beta\})$ is an orbit of $G_{\{\alpha,\beta\}}$. If Γ is contained in some G_α-block Δ_i then $G_{\alpha\beta}$ preserves Δ_i and so fixes the single point of $\Delta_i - \Gamma$; Lemma 1 now gives a contradiction. It follows that Γ intersects 3 of the G_α-blocks and that $G_{\alpha\beta}$ permutes $\{\Delta_2, \ldots, \Delta_q\}$ semi-regularly in orbits of size 3. From this it follows that $[G_{\alpha\beta} : K_\beta] = 3$ and that $G_{\alpha\beta}$ acts regularly on Γ and so acts regularly on $\Delta_1 - \{\beta\}$. Now H cannot act as S_4 on Δ_1 since $G_{\alpha\beta} = H_\beta$ acts regularly on $\Delta_1 - \{\beta\}$; thus H acts as A_4 on Δ_1 and hence K and K_β have normal Sylow 2-subgroups. The Sylow 2-subgroup of K_β is normal in $G_{\alpha\beta}$ and is a Sylow 2-subgroup of $G_{\alpha\beta}$ and it follows from Lemma 1 that K_β is an (elementary Abelian) 3-group. Now $K \lhd H$ and so K acts trivially or transitively on Δ_1; thus, either $K = 1$ or K acts transitively on each of $\Delta_1, \Delta_2, \ldots, \Delta_q$. In particular we have that either $K_\beta = 1$ so that $|G| = 12pq$ or that K acts primitively on each of $\Delta_1, \Delta_2, \ldots, \Delta_q$. In the latter case K acts faithfully on Δ_1; for, if N is the kernel of the action of K on Δ_1 and $N \neq 1$, then N acts transitively on some Δ_i contradicting the fact that $2^3 \nmid |K|$. Thus $|K_\beta| = 3$ and $|G| = 36pq$. In each of these cases, $|G| = 12pq$ or $|G| = 36pq$, we can obtain a contradiction by considering possibilities for $[G : N(P)]$ and using Sylow's theorem.

Case 3. 4 G_α-blocks of size q

In this case $Q \leqslant K$ and so K is transitive on each of Δ_1, Δ_2, Δ_3 and Δ_4. If N is the kernel of the action of K on Δ_1 and $N \neq 1$, then N acts transitively on some Δ_i which contradicts the fact that $q^2 \nmid |G|$. Thus K acts faithfully on each of Δ_1, Δ_2, Δ_3 and Δ_4. As in case 2 we consider the possibilities of G_α being soluble or insoluble separately.

If G_α is insoluble then H acts on Δ_1 as an insoluble group and so acts doubly transitively on Δ_1. Hence $\Gamma_1 = \Delta_1 - \{\beta\}$ is a $G_{\alpha\beta}$-orbit of size $q-1$. If Γ_1 is not $G_{\{\alpha,\beta\}}$-invariant then there exists another $G_{\alpha\beta}$-orbit Γ_2 of size $q-1$ such that $\Gamma_1 \cup \Gamma_2$ is an orbit of $G_{\{\alpha,\beta\}}$. By Lemma 3 there is yet another $G_{\alpha\beta}$-orbit Γ_3 of size $q-1$ and since it is a $G_{\{\alpha,\beta\}}$-orbit it is distinct from Γ_1 and Γ_2. If either of Γ_2 or Γ_3 is contained in any Δ_i then $G_{\alpha\beta}$ leaves Δ_i invariant and fixes the remaining point of Δ_i; using Lemma 1, this leads to a contradiction. It follows that, for at least one Δ_i, we have $0 < |\Gamma_2 \cap \Delta_i| < q-1$ and $0 < |\Gamma_3 \cap \Delta_i| < q-1$. But $\Gamma_2 \cap \Delta_i$ and $\Gamma_3 \cap \Delta_i$ are invariant under K_β and, as they are sets of imprimitivity for the action of $G_{\alpha\beta}$ on Γ_2 and Γ_3, we have $|\Gamma_2 \cap \Delta_i| \leqslant \frac{1}{2}(q-1)$ and $|\Gamma_3 \cap \Delta_i| \leqslant \frac{1}{2}(q-1)$. Consequently, K_β has at least 3 orbits on Δ_i. Now K acts doubly transitively on each of Δ_1 and Δ_i with characters $1+\chi_1$ and $1+\chi_2$, say, and the number of orbits of K_β on Δ_i is $(1+\chi_1, 1+\chi_2) \leqslant 2$ and this is a contradiction.

If G_α is soluble we shall show that $K_\beta = 1$. If $K_\beta \neq 1$ then K_β fixes precisely one point from each of Δ_1, Δ_2, Δ_3 and Δ_4 because K has a unique conjugacy class of subgroups of index q. Thus K_β and any conjugate of K_β fix exactly 5 points. Consider some conjugate $K_\beta{}^g$ of K contained in $G_{\alpha\beta}$. If $K_\beta{}^g \nleqslant K$ then some Δ_i contains none of the fixed points of $K_\beta{}^g$ and hence there is some Δ_j which contains at least two of these fixed points; but then $K_\beta{}^g$ must fix pointwise the whole of Δ_j and so has more than 5 fixed points. Thus $K_\beta{}^g \leqslant K$ and, as K has a unique subgroup of index q which fixes β, we have $K_\beta{}^g = K_\beta$. Therefore K_β is weakly closed in $G_{\alpha\beta}$ and Lemma 1 gives us a contradiction. This means that $K_\beta = 1$ as we asserted. Hence $|G| = vpq$ where $v|24$ and we can obtain a contradiction by considering possibilities for $[G:N(P)]$ and using Sylow's theorem.

Case 4. 2 G_α-blocks of size $2q$

In this case we have $H = K$ and we may assume, by case 3, that K acts primitively on each of Δ_1 and Δ_2. By the same argument as in case 3, it also acts faithfully on Δ_1 and Δ_2. If K acts doubly transitively on Δ_1 then $\Delta_1 - \{\beta\}$ is an orbit of $G_{\alpha\beta}$ of size $2q-1$. Lemma 2 implies that $\Delta_1 - \{\beta\}$ is not $G_{\{\alpha,\beta\}}$-invariant and hence there is another $G_{\alpha\beta}$-orbit of size $2q-1$. It follows that $G_{\alpha\beta}$ fixes a third point and therefore Lemma 1 can be applied to gain a contradiction.

If K does not act doubly transitively on Δ_1 then a theorem of H. Wielandt **[11]** tells us that K has rank 3 on Δ_1 (and on Δ_2), $2q = m^2+1$ for some integer m, and the orbits of $G_{\alpha\beta} = K_\beta$ on Δ_1 are $\Gamma_0 = \{\beta\}$, Γ_1 and Γ_2 of sizes 1, $\frac{1}{2}m(m-1)$ and $\frac{1}{2}m(m+1)$ respectively. Let $\pi_1 = 1+\psi_1+\chi_1$ and $\pi_2 = 1+\psi_2+\chi_2$ be the decompositions of the permutation characters of K corresponding to the actions of K on Δ_1 and Δ_2. Since $(\pi_1, \pi_2) = 1$, 2 or 3, $G_{\alpha\beta}$ has 1, 2 or 3 orbits on Δ_2.

If $G_{\alpha\beta}$ has only one orbit on Δ_2 then $\Delta_1 - \{\beta\}$ is the only $G_{\alpha\beta}$-invariant set of size $2q-1$. Hence $\Delta_1 - \{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant and Lemma 2 provides a contradiction.

If $G_{\alpha\beta}$ has two orbits Γ_3 and Γ_4 on Δ_2 then again Lemma 2 applies unless at least one of Γ_1 and Γ_2 is not $G_{\{\alpha,\beta\}}$-invariant. Suppose that Γ_1 is not $G_{\{\alpha,\beta\}}$-invariant. Then we may suppose that $|\Gamma_3| = |\Gamma_1|$ and that $\Gamma_1 \cup \Gamma_3$ is a $G_{\{\alpha,\beta\}}$-orbit. Therefore $|\Gamma_3| = \frac{1}{2}m(m-1)$, $|\Gamma_4| = \frac{1}{2}m(m+1)$ and the orbits of $G_{\{\alpha,\beta\}}$ are $\{\alpha, \beta\}$, Γ_2, $\Gamma_1 \cup \Gamma_3$ and Γ_4. Clearly, $G_{\{\alpha,\beta\}}$ does not have an invariant set of size $2q-1$ and this contradicts Lemma 3. A similar contradiction arises from assuming that Γ_2 is not $G_{\{\alpha,\beta\}}$-invariant.

Finally, we have to consider the case in which $G_{\alpha\beta}$ has 3 orbits on Δ_2. We may assume in any case that $|N(P)| = pq$ since in all other cases $|N(P)|$ is even and another theorem of P. M. Neumann, **[8]**, would yield that G is triply transitive. The theorem of P. M. Neumann stated earlier tells us that $G_{\alpha\beta}$ has at most $(p-1)/q = 4$ orbits in $\Omega - \{\alpha, \beta\}$ and this contradicts the fact that $G_{\alpha\beta}$ has 3 orbits on Δ_2 and so 5 orbits on $\Omega - \{\alpha, \beta\}$.

This completes the proof of Theorem A. We note that the primality of p was used heavily in the last paragraph. If this case could be dealt with by other means then it would probably be possible to prove a result about doubly transitive groups of degree $4q+1$ somewhat on the lines of Theorem B.

Proof of Theorem B

Assume for a contradiction that there exists a group G which satisfies the conditions but not the conclusions of Theorem B. By a search through the list of groups of degrees 7 and 10 (see **[9]**) we see that there are no counterexamples to the theorem with $q \leqslant 3$ and so we may assume that $q \geqslant 5$. As an easy consequence of Lemma 4 we have that G is not a group of automorphisms of a non-trivial block design on Ω with $\lambda = 1$. We also observe that G cannot be a Zassenhaus group, for these have degree p^a+1 or 2^p, where p is prime **[2, 6** and **10]**, and $3q+1$ is of this form only if $q = 3$.

Let Q be a Sylow q-subgroup of G_α, for some $\alpha \in \Omega$, and let $\Delta_1, \Delta_2, \ldots$ be a non-trivial system of imprimitivity for the action of G_α on $\Omega - \{\alpha\}$. Let

$$\beta \in \Delta_1, \quad H = \{x \in G_\alpha \mid \Delta_1 x = \Delta_1\} \quad \text{and} \quad K = \{x \in G_\alpha \mid \Delta_i x = \Delta_i, \quad i = 1, 2, \ldots\}.$$

Then $G_{\alpha\beta} \leqslant H$ and $K \lhd G$. There are two cases to consider.

Case 1. q G_α-blocks of size 3

In this case an argument similar to that of case 2 of Theorem A can be used. Instead, however, we give the following argument of G. Higman which avoids the assumption that q is prime. We may assume that $G_{\alpha\beta}$ fixes no points other than α and β by Lemma 1 and it follows from this that $N(G_{\alpha\beta}) = G_{\{\alpha, \beta\}}$. If we let $\Delta_1 = \{\beta, \gamma, \delta\}$ we have that $\{\gamma, \delta\}$ is $G_{\alpha\beta}$-invariant and hence $G_{\alpha\beta}$ is contained in $G_{\{\gamma, \delta\}}$ as a subgroup of index 2. It follows that $G_{\{\gamma, \delta\}} \leqslant N(G_{\alpha\beta})$ and so $G_{\{\alpha, \beta\}} = G_{\{\gamma, \delta\}}$. But now Lemma 2 provides a contradiction.

Case 2. 3 G_α-blocks of size q

As $Q \leqslant K$, K is transitive on each of Δ_1, Δ_2 and Δ_3 and, exactly as in the proof of case 3 of Theorem A, K acts faithfully on each of Δ_1, Δ_2 and Δ_3.

If G_α is insoluble we may reach a contradiction by arguing as in case 3 of Theorem A. If G_α is soluble an argument similar to that of case 3 of Theorem A may be used to show that $K_\beta = 1$, the only difference being that we use fixed-point sets of order 4 rather than sets of order 5. Hence we have $|G_{\alpha\beta}| = 2$ and, as G is not a Zassenhaus group, $G_{\alpha\beta}$ fixes a third point, contradicting Lemma 1.

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