SPACES OF MATRICES WITH SEVERAL ZERO EIGENVALUES

M. D. ATKINSON

Let V be an n-dimensional vector space over some field F, $|F| \geqslant n$, and let $\mathscr{X}$ be a space of linear mappings from V into itself ($\mathscr{X} \leqslant \operatorname{Hom}(V, V)$) with the property that every mapping has at least r zero eigenvalues. If $r = 0$ this condition is vacuous but if $r = 1$ it states that $\mathscr{X}$ is a space of singular mappings; in this case Flanders [1] has shown that $\dim \mathscr{X} \leqslant n(n-1)$ and that if $\dim \mathscr{X} = n(n-1)$ then either $\mathscr{X} = \operatorname{Hom}(V, U)$ for some $(n-1)$-dimensional subspace U of V or there exists $v \in V, v \neq 0$ such that $\mathscr{X} = \{X \in \operatorname{Hom}(V, V) : vX = 0\}$. At the other extreme $r = n$, the case of nilpotent mappings, a theorem of Gerstenhaber [2] states that $\dim \mathscr{X} \leqslant \frac{1}{2}n(n-1)$ and that if $\dim \mathscr{X} = \frac{1}{2}n(n-1)$ then $\mathscr{X}$ is the full algebra of strictly lower triangular matrices (with respect to some suitable basis of V).

In this paper we intend to derive similar results for a general value of r thereby providing some common ground for the above theorems. The condition on the cardinal of F plays an important role in our proofs although we do not know whether it is essential to the theorems. Flanders required also the constraint char $F \neq 2$ at one point in his proof; we do not require this and in fact our type of argument allows the field in Flanders' theorem to have arbitrary characteristic.

THEOREM 1. *Let V be an n-dimensional vector space over some field F, $|F| \geqslant n$, and let $\mathscr{X}$ be a subspace of* $\operatorname{Hom}(V, V)$. *Suppose that every mapping in $\mathscr{X}$ has at least r zero eigenvalues. Then* $\dim \mathscr{X} \leqslant \frac{1}{2}r(r-1) + n(n-r)$.

Proof. Gerstenhaber's theorem proves our result in the case $n - r = 0$ and provides the first step of an induction on $n-r$. Assume now that $n > r > 0$ and that the theorem holds for spaces whose transformations all have more than r zero eigenvalues. Since $\frac{1}{2}r(r+1) + n(n-r-1) < \frac{1}{2}r(r-1) + n(n-r)$ we can suppose that $\mathscr{X}$ contains a transformation with precisely r zero eigenvalues. By an appropriate choice of basis for V we may take this to be (represented by)

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

with A_1 an $r \times r$ nilpotent matrix and A_2 non-singular. By taking A_1 in rational canonical form we may take it to be strictly lower triangular. From now on we shall take $\mathscr{X}$ to consist of $n \times n$ matrices $\begin{pmatrix} J & K \\ M & N \end{pmatrix}$ partitioned so that J is an $r \times r$ matrix.

Consider any matrix $B \in \mathscr{X}$ of the form $\begin{pmatrix} J & K \\ 0 & N \end{pmatrix}$. For all $\alpha \in F$ $\begin{pmatrix} J + \alpha A_1 & K \\ 0 & N + \alpha A_2 \end{pmatrix}$ has at least r zero eigenvalues and, except for at most $n-r$ eigenvalues of $-A_2^{-1}N$, $N + \alpha A_2$ is non-singular. Hence for all but at most $n-r$ values of α, $J + \alpha A_1$ is

Received 24 May, 1979.

[BULL. LONDON MATH. SOC., 12 (1980), 89–95]

nilpotent. However the coefficients of the characteristic polynomial of $J+\alpha A_1$ are polynomials in α which, since A_1 is singular, are of degree at most $r-1$. The non-leading coefficients of the characteristic polynomial all vanish for at least r values of α (since $|F| \geqslant n$) and so are identically zero i.e. $J+\alpha A_1$ is nilpotent for all α and in particular J is nilpotent. Let $\mathscr{J}$ be the space of nilpotent $r \times r$ matrices which arise in this way; then $\dim \mathscr{J} \leqslant \frac{1}{2}r(r-1)$.

Let $J_1, \ldots, J_t$ be a basis for $\mathscr{J}$ and consider the r^2 t-vectors l_{pq} whose k-th component is $(J_k)_{pq}$. Since $t \leqslant \frac{1}{2}r(r-1)$ we may choose a set L of $\frac{1}{2}r(r-1)$ index pairs (p, q) such that $\{l_{pq} : (p, q) \in L\}$ spans the space generated by all the l_{pq}. Let U be the complementary set of $\frac{1}{2}r(r+1)$ index pairs. Then every $r \times r$ matrix J may be written uniquely as $J = J_L + J_U$ where $(J_L)_{pq} = 0$ if $(p, q) \in U$ and $(J_U)_{pq} = 0$ if $(p, q) \in L$. By the choice of L we have the condition $J \in \mathscr{J}$ and $J_L = 0$ implies $J = 0$, or equivalently

$$\begin{pmatrix} J_L + J_U & K \\ 0 & N \end{pmatrix} \in \mathscr{X} \quad \text{and} \quad J_L = 0 \quad \text{implies} \quad J_U = 0. \tag{*}$$

By expressing each matrix of $\mathscr{X}$ in the form $\begin{pmatrix} J_L + J_U & K \\ M & N \end{pmatrix}$ we may define a linear map ϕ on $\mathscr{X}$ by

$$\begin{pmatrix} J_L + J_U & K \\ M & N \end{pmatrix} \xrightarrow{\ \phi\ } \begin{pmatrix} J_L & 0 \\ M & N \end{pmatrix}.$$

Then, by (*), $\left\{\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \in \mathscr{X}\right\}$ is the kernel of ϕ. Let

$$\mathscr{K} = \left\{K : \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \in \mathscr{X}\right\}, \quad \mathscr{L} = \left\{J_L : \begin{pmatrix} J & K \\ M & N \end{pmatrix} \in \mathscr{X}\right\},$$

$$\mathscr{M} = \left\{M : \begin{pmatrix} J & K \\ M & N \end{pmatrix} \in \mathscr{X}\right\}, \quad \mathscr{N} = \left\{N : \begin{pmatrix} J & K \\ M & N \end{pmatrix} \in \mathscr{X}\right\}.$$

$$\begin{aligned} \text{Clearly } \dim \mathscr{X} &= \dim \text{range}(\phi) + \dim \text{kernel}(\phi) \\ &\leqslant \dim \mathscr{L} + \dim \mathscr{M} + \dim \mathscr{N} + \dim \mathscr{K} \\ &\leqslant \tfrac{1}{2}r(r-1) + (n-r)^2 + \dim \mathscr{M} + \dim \mathscr{K} \end{aligned}$$

and the proof will be completed when we have shown that $\dim \mathscr{M} + \dim \mathscr{K} \leqslant r(n-r)$. To do this we shall construct a non-singular bilinear form $\theta : \mathscr{A}_{r,n-r} \times \mathscr{A}_{n-r,r} \to F$ where $\mathscr{A}_{r,n-r}$, $\mathscr{A}_{n-r,r}$ denote the full spaces of $r \times (n-r)$, $(n-r) \times r$ matrices over F and show that θ vanishes on $\mathscr{K} \times \mathscr{M}$. This will prove that $\mathscr{M}$ is contained in the annihilator of $\mathscr{K}$ which has dimension $r(n-r) - \dim \mathscr{K}$, giving $\dim \mathscr{M} \leqslant r(n-r) - \dim \mathscr{K}$ as required.

Consider any matrices $K \in \mathscr{K}$, $M \in \mathscr{M}$. Then there exist $Y = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} = (y_{ij}) \in \mathscr{X}$ and $X = \begin{pmatrix} J & H \\ M & N \end{pmatrix} = (x_{ij}) \in \mathscr{X}$. Since $X + \alpha A + \beta Y$ has at least r zero eigenvalues for all

α, β the coefficient of $\alpha^{n-r-1}\beta\lambda^{r-1}$ in $\det(X+\alpha A+\beta Y-\lambda I)$ is zero (see [2] p. 615). The total coefficient of λ^{r-1} in this determinant is (apart from a sign) the sum of all $(n-r+1)$-rowed principal minors of $X+\alpha A+\beta Y$ and we require the coefficient of $\alpha^{n-r-1}\beta$ in this sum. Consider a typical one of these minors

$$M(S,T) = \det\begin{pmatrix} J'+\alpha A'_1 & H'+\beta K' \\ M' & N'+\alpha A'_2 \end{pmatrix}$$

where $J'+\alpha A'_1$ is the principal submatrix of $X+\alpha A+\beta Y$ whose rows are indexed by the set $S \subseteq \{1,2,\dots,r\}$ and $N'+\alpha A'_2$ is the principal submatrix whose rows are indexed by the set $T \subseteq \{r+1,\dots,n\}$ and where $|S \cup T| = n-r+1$. By direct calculation the coefficient of $\alpha^{n-r-1}\beta$ in this minor is, to within a sign,

$$\sum_{\substack{i,j\in S\\ s,t\in T}} C_{ij}D_{st}y_{it}x_{sj}$$

where (C_{ij}) is the matrix of cofactors of A'_1 and (D_{st}) is the matrix of cofactors of A'_2. We shall express such a coefficient as

$$yZx^t$$

where

$$y = (y_{1,r+1},\dots,y_{1n},y_{2,r+1},\dots,y_{2n},\dots,y_{r,r+1},\dots,y_{rn}),$$

$$x = (x_{r+1,1},\dots,x_{n1},x_{r+1,2},\dots,x_{n2},\dots,x_{r+1,r},\dots,x_{nr}),$$

and Z is an $r(n-r)\times r(n-r)$ matrix whose entries depend upon (C_{ij}) and (D_{st}).

If $|S| > 1$ then, since A'_1 is strictly lower triangular, $C_{ij} = 0$ for $i,j \in S$, $i \geqslant j$ and the matrix Z then has the form

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

with zeros above the diagonal and where each diagonal 0 represents an $(n-r)\times(n-r)$ matrix of zeros.

If $|S| = 1$, say $S = \{i\}$, then $|T| = n-r$, $C_{ii} = 1$ and $(D_{st}) = E$ is the matrix of cofactors of A_2. In this case the matrix Z has the form

$$\begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & E & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

where each diagonal block is an $(n-r)\times(n-r)$ matrix, E is the i-th diagonal block and there are zeros elsewhere.

Summing over all appropriate S, T we obtain the coefficient of $\alpha^{n-r-1}\beta\lambda^{r-1}$ in the form yZx^t where

$$Z = \begin{pmatrix} E & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix}$$

and has zeros above the diagonal. Since A_2 is non-singular so also is E and thus Z is non-singular. This matrix Z defines our non-singular bilinear form vanishing on $\mathcal{K}\times\mathcal{M}$ and so completes the proof of the theorem.

In the next theorem we consider the case where dim $\mathcal{X}$ is maximal and determine the possibilities for $\mathcal{X}$. We first prove

LEMMA. *Let V be an n-dimensional vector space over some field F, $|F|\geqslant n$, and let $\mathcal{X}$ be a subspace of* Hom (V, V). *Suppose that W_1 and W_2 are $\mathcal{X}$-invariant subspaces of V such that*

(i) dim $W_1 = s_1$, dim $V/W_2 = s_2$, dim $W_1 \cap W_2 = t$,

(ii) *$\mathcal{X}$ induces spaces of nilpotent mappings in each of W_1 and V/W_2,*

(iii) *every transformation in $\mathcal{X}$ has at least r zero eigenvalues.*

Then $\dim\mathcal{X}\leqslant \frac{1}{2}r(r-1)+n(n-r)-(s_1-t)(n-s_2-t)$.

Proof. W_1+W_2 has dimension $\dim W_1+\dim W_2-\dim W_1\cap W_2 = s_1+n-s_2-t$ and we may choose a basis of it having the form

$$e_1,\ldots, e_t, f_1,\ldots, f_{s_1-t}, g_1, \ldots g_{n-s_2-t}$$

such that $e_1,\ldots, e_t$ is a basis for $W_1\cap W_2$, $e_1,\ldots, e_t$, $f_1,\ldots, f_{s_1-t}$ is a basis for W_1, and $e_1,\ldots, e_t, g_1,\ldots, g_{n-s_2-t}$ is a basis for W_2. We complete this to a basis of V with $h_1,\ldots, h_{s_2-s_1+t}$.

With respect to this basis each transformation of $\mathcal{X}$ has matrix

$$\begin{pmatrix} N_1 & 0 & 0 & 0 \\ A_1 & N_2 & 0 & 0 \\ A_2 & 0 & B & 0 \\ A_3 & A_4 & A_5 & N_3 \end{pmatrix}.$$

N_1 and N_2 are nilpotent $t\times t$ and $(s_1-t)\times(s_1-t)$ matrices since $\mathcal{X}$ restricted to W_1 is a space of nilpotent mappings. N_3 is a nilpotent $(s_2-s_1+t)\times(s_2-s_1+t)$ matrix since $\mathcal{X}$ induces nilpotent transformations on $V/(W_1+W_2)$. Since the whole matrix has at least r zero eigenvalues the $(n-s_2-t)\times(n-s_2-t)$ matrix B has at least $r-s_2-t$ zero eigenvalues. Now, by Theorem 1, the space of all $\begin{pmatrix} N_1 & 0 \\ A_1 & N_2 \end{pmatrix}$ has dimension at most $\frac{1}{2}s_1(s_1-1)$, the space of all matrices N_3 has dimension at most

$$\tfrac{1}{2}(s_2-s_1+t)(s_2-s_1+t-1),$$

and the space of all matrices B has dimension at most

$$\tfrac{1}{2}(r-s_2-t)(r-s_2-t-1)+(n-s_2-t)(n-r).$$

A routine calculation now shows that

$$\dim \mathscr{X} \leqslant \tfrac{1}{2}r(r-1)+n(n-r)-(s_1-t)(n-s_2-t).$$

THEOREM 2. *Let V be an n-dimensional vector space over some field F, $|F| \geqslant n$, and let $\mathscr{X}$ be a subspace of* Hom (V, V). *Suppose that every mapping in $\mathscr{X}$ has at least r zero eigenvalues and that $\mathscr{X}$ has dimension precisely $\frac{1}{2}r(r-1)+n(n-r)$. Then V contains $\mathscr{X}$-invariant subspaces V_1, V_2 with $V_1 \leqslant V_2$ and* $\dim V_1 + \dim V/V_2 = r$ *and such that $\mathscr{X}$ operates on each of V_1 and V/V_2 as the full algebra of strictly lower triangular matrices (with respect to suitable bases). Moreover the subspaces V_1, V_2 are uniquely determined by $\mathscr{X}$ unless $r = n$.*

Proof. We continue to use the machinery and notation $A, L, U, \phi, \mathscr{L}, \mathscr{M}, \mathscr{N}, \mathscr{K}, \theta$ introduced in the proof of Theorem 1.

From

$$\begin{aligned}\tfrac{1}{2}r(r-1)+n(n-r) = \dim \mathscr{X} &\leqslant \dim \mathscr{L} + \dim \mathscr{N} + \dim \mathscr{M} + \dim \mathscr{K} \\ &\leqslant \tfrac{1}{2}r(r-1)+(n-r)^2+\dim \mathscr{M} + \dim \mathscr{K} \\ &\leqslant \tfrac{1}{2}r(r-1)+(n-r)^2+r(n-r) = \tfrac{1}{2}r(r-1)+n(n-r)\end{aligned}$$

we obtain

$$\dim \mathscr{L} = \tfrac{1}{2}r(r-1),\ \dim \mathscr{N} = (n-r)^2,\ \dim \mathscr{M} + \dim \mathscr{K} = r(n-r)$$

and $\mathscr{M}, \mathscr{K}$ are the full annihilators of each other with respect to the bilinear form θ. Moreover, identifying $\mathscr{L}$ with $\left\{\begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} : L \in \mathscr{L}\right\}$ etc., we have range(ϕ) $= \mathscr{L} \oplus \mathscr{M} \oplus \mathscr{N}$.

In particular range(ϕ) contains a matrix $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and so $\mathscr{X}$ contains a matrix $\begin{pmatrix} 0 & K \\ 0 & I \end{pmatrix}$. This latter matrix is similar to $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and so we could have taken this matrix as our matrix A to begin with. Let us suppose that this had been done. With this choice of A the bilinear form $\theta(U, V)$ is easily seen to be $\operatorname{tr}(UV)$.

Now let $X = \begin{pmatrix} H & J \\ M & N \end{pmatrix}$ be any matrix in $\mathscr{X}$ and consider the characteristic polynomial $\det(X+\alpha A-\lambda I)$. In this polynomial the coefficients of

$\alpha^{n-r}, \alpha^{n-r}\lambda, \ldots, \alpha^{n-r}\lambda^{r-1}$ are all zero and are plainly equal to the coefficients of $\lambda^0, \lambda, \ldots, \lambda^{r-1}$ in $\det(H-\lambda I)$. Thus H is nilpotent. Let $\mathscr{H}$ be the space of matrices H arising in this way. Then $\frac{1}{2}r(r-1) = \dim \mathscr{L} \leqslant \dim \mathscr{H} \leqslant \frac{1}{2}r(r-1)$ by Gerstenhaber's theorem and so $\dim \mathscr{H} = \frac{1}{2}r(r-1)$. By Gerstenhaber's theorem again $\mathscr{H}$ may be reduced to the full algebra of strictly lower triangular matrices with a similarity matrix T. Therefore by transforming $\mathscr{X}$ by the similarity matrix $\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}$ we can take $\mathscr{H}$ to be the full algebra of strictly lower triangular matrices. Having done this we may compute the coefficient of $\alpha^{n-r-1}\lambda^{r-1}$ in $\det(X+\alpha A-\lambda I)$. This yields $\operatorname{tr}(JM) = 0$.

Consider any matrix $Y = \begin{pmatrix} H & K \\ 0 & N \end{pmatrix}$ in $\mathscr{X}$. Then for all $M \in \mathscr{M}$ there exists $Z = \begin{pmatrix} 0 & J \\ M & 0 \end{pmatrix} \in \mathscr{X}$ and we have $\operatorname{tr}(JM) = 0$ and, since $Y+Z \in \mathscr{X}$, $\operatorname{tr}\big((K+J)M\big) = 0$. Hence $\operatorname{tr}(KM) = 0$ and K is in the annihilator of $\mathscr{M}$ with respect to θ i.e. $K \in \mathscr{K}$. Hence $\mathscr{X}$ contains $\begin{pmatrix} H & 0 \\ 0 & N \end{pmatrix}$ for all strictly lower triangular H and all N.

Now we show that either every matrix in $\mathscr{X}$ has first row zero or that every matrix in $\mathscr{X}$ has r-th column zero. Suppose that this is not so, in which case we may find some matrix $X = (x_{ij})$ which has neither first row nor r-th column zero, say $x_{1j}x_{ir} \neq 0$ for some $i, j > r$. As shown above $\mathscr{X}$ contains also the matrix $\begin{pmatrix} H & 0 \\ 0 & N \end{pmatrix}$ where H has ones in the $(t+1, t)$ position, $t = 1, 2, \ldots, r$, and zeros elsewhere and N is any matrix with a single one in every row and column except for the i-th row and j-th column and zeros elsewhere. But then $x_{1j}x_{ir}$ is the coefficient of α^{n-2} in $\det(X+\alpha W)$ and so is zero since $r \geqslant 1$, a contradiction.

In terms of matrices the existence part of the theorem asserts that the matrices of $\mathscr{X}$ are simultaneously similar to matrices

$$\begin{pmatrix} L_1 & 0 & 0 \\ A & L_2 & B \\ C & 0 & D \end{pmatrix}$$

where L_1, L_2 are strictly lower triangular matrices of degrees r_1, r_2 with $r_1+r_2 = r$. We can now prove this statement by induction taking as an inductive hypothesis that the statement is true for smaller values of n and all appropriate values of r.

In the case where the matrices of $\mathscr{X}$ have zero first row we consider the space of $(n-1)\times(n-1)$ matrices $\bar{\mathscr{X}}$ obtained by deleting the first row and first column. It is clear that each matrix of $\bar{\mathscr{X}}$ has at least $r-1$ zero eigenvalues and hence $\dim \bar{\mathscr{X}} \leqslant \frac{1}{2}(r-1)(r-2)+(n-1)(n-r)$. On the other hand from the definition of $\bar{\mathscr{X}}$, $\dim \bar{\mathscr{X}} \geqslant \dim \mathscr{X}-(n-1) = \frac{1}{2}(r-1)(r-2)+(n-1)(n-r)$. Thus equality holds and the inductive hypothesis can be applied to $\bar{\mathscr{X}}$. Hence, for some non-singular $(n-1)\times(n-1)$ matrix T the matrices of $T^{-1}\bar{\mathscr{X}}T$ have the form

$$\begin{pmatrix} L_1' & 0 & 0 \\ A & L_2 & B \\ C & 0 & D \end{pmatrix}$$

with L_1', L_2 strictly lower triangular of degree r_1-1, r_2 and $r_1-1+r_2 = r-1$. But then taking $S = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$ the matrices of $S^{-1}\mathscr{X}S$ have the required form.

In the case where the matrices of $\mathscr{X}$ have zero r-th column we use a similar argument taking $\bar{\mathscr{X}}$ to be the space obtained by deleting the r-th row and r-th column from the matrices of $\mathscr{X}$.

Finally, returning to the language of vector spaces and linear mappings, suppose that U_1, U_2 are two subspaces of V which have all the stated properties of V_1, V_2. Let

$$W_1 = U_1+V_1, W_2 = U_2 \cap V_2, \dim W_1 = s_1, \dim V/W_2 = s_2, \dim W_1 \cap W_2 = t.$$

Then $\mathscr{X}$ induces nilpotent mappings in both W_1 and V/W_2 and therefore $\mathscr{X}$, W_1, W_2 satisfy all the conditions of the lemma. It follows that

$$\tfrac{1}{2}r(r-1)+n(n-r) \leqslant \tfrac{1}{2}r(r-1)+n(n-r)-(s_1-t)(n-s_2-t).$$

But $(s_1-t)(n-s_2-t)$ is non-negative and so is zero. Hence $s_1 = t$ or $n = s_2+t$. From $s_1 = t$ it follows that $W_1 \leqslant W_2$ i.e. $U_1+V_1 \leqslant U_2 \cap V_2$ and hence $U_1 = V_1, U_2 = V_2$ since otherwise all transformations in $\mathscr{X}$ would have more than r zero eigenvalues, contradicting Theorem 1. From $n = s_2+t$ it follows that $W_2 \leqslant W_1$ and $\mathscr{X}$ is a space of nilpotent matrices i.e. $r = n$.

References

1. H. Flanders, "On spaces of linear transformations with bounded rank", *J. London Math. Soc.*, 37 (1962), 10–16.
2. M. Gerstenhaber, "On nilalgebras and linear varieties of nilpotent matrices I", *Amer. J. Math.*, 80 (1958), 614–622.

Department of Computing Mathematics,
University College,
Cardiff.