SPACES OF LINEAR TRANSFORMATIONS
OF EQUAL RANK

by

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Abstract: Examples are obtained for infinite sets of inequivalent spaces (of the same dimension) of linear transformations of equal rank in a space $L(V,W)$. This is done for both decomposable and indecomposable spaces.

§1 INTRODUCTION

This paper is concerned with <u>k-spaces</u> of linear transformations from one (finite dimensional) vector space V into another space W, that is subspaces of $L(V,W)$ which, apart from the zero mapping, contain only transformations of rank k. Two k-spaces $H_1 \leq L(V_1,W_1)$ and $H_2 \leq L(V_2,W_2)$ are said to be <u>equivalent</u> if there exist isomorphisms $\alpha : V_1 \to V_2$, $\beta : W_1 \to W_2$, $\gamma : H_1 \to H_2$ such that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\ h\ } & W_1 \\ \alpha \downarrow & & \downarrow \beta \\ V_2 & \xrightarrow{\ \gamma(h)\ } & W_2 \end{array}$$

commutes, for all $h \in H_1$. Equivalent spaces share all the same algebraic properties and are usually regarded as being 'the same space.' Our aim here is to show that the number of inequivalent k-subspaces of $L(V,W)$ can be infinite.

The study of k-spaces was begun in [2] where a k-space $H \leq L(V,W)$ was termed <u>(essentially) decomposable</u> if V had a direct sum decomposition $V = V_1 \oplus V_2$ where $\dim V_1 = i$ and $\dim(\langle HV_2 \rangle) = j$ with $i+j = k$. The decomposable spaces are the simplest types of k-spaces and several of the results of [2] gave sufficient conditions for a k-space to be decomposable. A more recent sufficient condition for decomposability is $\dim H > k+1$ [1]

Of course this subject can be studied in the language of matrices by choosing bases for V and W and representing each linear transformation by its matrix. The notion of equivalence between two k-spaces H_1 and H_2 of $m\times n$ matrices is just that

$$H_2 = pH_1q = \{phq : h \in H_1\}$$

for non-singular $m\times m$ and $n\times n$ matrices p and q. The matrices p and q correspond respectively to a general row operation and a general column operation performed simultaneously on the matrices of H_1. For a k-space of matrices to be decomposable means that some space equivalent to it has all its matrices of the form

$$\left[\begin{array}{c|c} i\times j & \\ \hline & 0 \end{array}\right]$$

with $i+j = k$. We shall find use for both ways of representing k-spaces.

Fairly complete information is available for small values of k. It is rather easy to show that all 1-spaces are decomposable. For 2-spaces it can be shown that, apart from trivial variations, the 3-dimensional space of all 3×3 skew-symmetric matrices is the only indecomposable example. Moreover using Kronecker's theory of pencils together with some of the ideas in section 3 of this paper one can prove that, for fixed $m = \dim V$ and

$n = \dim W$, there are only a finite number of decomposable 2-spaces. In the case of 3-spaces unpublished work of Atkinson and Lloyd demonstrates that there are precisely 8 different indecomposable examples.

However this behaviour does not persist for larger values of k :- for some values of m,n,k there can be an infinite number of k-spaces of $m \times n$ matrices all having the same dimension. We give two examples of this, in section 2 of indecomposable spaces, and in section 3 of decomposable spaces. The examples are described in more generality than strictly necessary in order to better illustrate the techniques, which we feel should have wider applicability.

Throughout the paper we shall take the ground field F to be the field of complex numbers but most of the theory carries over to arbitrary fields.

§2 Indecomposable Spaces

The main purpose of this section is to give an example of an infinite set of inequivalent indecomposable k-subspaces of equal dimension in a particular space, $L(V,W)$. We begin by considering the linear maps from $\Lambda^r U$ to $\Lambda^{r+1} U$ obtained by exterior multiplication of $\Lambda^r U$ by a vector in U . U is an n-dimensional vector space over the complex number field F and r is an integer, $1 \leq r < n$. For an f to be in H_r means that there is an $x \in U$ such that for all $X \in \Lambda^r U$ we have $f(X) = x \wedge X$. The kernel of f is $x \wedge \Lambda^{r-1} U$ which, for nonzero x , has dimension $\binom{n-1}{r-1}$. The rank of f is $\binom{n}{r} - \binom{n-1}{r-1} = \binom{n-1}{r}$. We let $k = \binom{n-1}{r}$ and note that H_r is a k-subspace of $L(\Lambda^r U , \Lambda^{r+1} U)$. We prove

<u>2.1 Theorem:</u> H_r <u>is indecomposable if</u> $1 \leq r < n-1$.

<u>Proof</u>: Suppose $\Lambda^r U = V_1 \oplus V_2$ with $\dim V_1 = i$, $\dim < H_r V_2 > = j$, and $i + j = k$. Certainly $i = 0$ is impossible for it implies

$$\binom{n-1}{r} = \dim(\Lambda^{r+1} U) = \binom{n}{r+1}$$

contradicting $r < n-1$. For nonzero $h \in H_r$, the kernel of h is contained in V_2 . These kernels all have the form $x \wedge \Lambda^{r-1} U$ and since they generate all $\Lambda^r U$ they cannot all be contained a proper subspace.

In general one can form k-subspaces from a given k-subspace H of $L(V,W)$ as follows. Let $W' \subseteq W$ be a supspace for which $h(V) \cap W' = \{0\}$ for every $h \in H$ and let $\eta : W \to W/W'$ be the natural map. The subspace

$$\eta H = \{\eta \circ h : h \quad H\}$$

is a k-subspace of $L(V,W/W')$. We prove

2.2 Theorem: H is decomposable if and only if ηH is decomposable.

It is clear that it suffices to prove this theorem for the case when $\dim W' = 1$. We begin by first proving

2.3 Lemma. Let $H \subseteq L(V,W)$ be a k-subspace and suppose $\dim W = k+1$. Then HV is a subspace of W .

Proof: If hV is a fixed subspace of W as h varies over non-zero elements of H then HV is this subspace. Suppose then that H contains two independent elements f and g such that $fV \neq gV$. Let $H' = < \{f,g\} >$. Let

$$V = V_o \oplus \cdots \oplus V_r \quad \text{and}$$
$$W = W_o \oplus \cdots \oplus W_r$$

be the decompositions for H' given by 3.1 of [2] . If $W_0 \neq 0$ then $\dim W_0 = 1$ and $HV = W_1 \oplus \cdots \oplus W_r$ is a k-dimensional subspace of W . Therefore $W_0 = \{0\}$. Then there is precisely one index i for which $\dim W_i = 1 + \dim V_i$, because no such i implies $W_0 \neq \{0\}$ while more than one such i implies $\dim W \geq k+2$, neither of which is possible. For definiteness we suppose $i = 1$ so that $\dim W_1 = 1 + \dim V_1$. Let $\omega \in W$, $\omega = \omega_1 + \cdots + \omega_r$ where $\omega_i \in W_i$, $i = 1, \ldots, r$. By 3.2 in [2] it is clear that we can find $a, b \in F$ and $v_1 \in V_1$ such that $(af + bg)v_1 = \omega_1$. For the remaining $i = 2, \ldots, r$, $af + bg$ maps V_i onto W_i and so there are $v_i \in V_i$ such that $(af + bg)v_i = \omega_i$. The lemma follows.

We now pass on to the proof of the theorem.

If H is decomposable then so is ηH because the decomposition of V for H will suffice for ηH.

Let us suppose ηH is decomposable and let

$$V = V_0 \oplus V_1$$

where $\dim V_0 = i$ and $\dim \langle \eta H V_1 \rangle = k - i$. Since ηH restricted to V_0 is an i-subspace the same is true of H. We note that

$$(\eta h V_0) \cap (\eta h V_1) = 0$$

for nonzero $h \in H$. Therefore $(hV_0) \cap (hV_1)$ is zero or W'. But hV_0 cannot contain a nonzero element of W' because $\eta h | V_0$ has rank equal to $\dim U_0$. Therefore $\rho(h | V_1) = k - i$ for nonzero $h \in H$ and

$$HV_1 \subseteq \eta H V_1 + W' ,$$

a $(k - i + 1)$ - dimensional space. By the lemma HV_1 is a subspace. Either HV_1 has dimension $k - i$, in which case H is decomposable or

$$HV_1 = \eta H V_1 + W' \quad .$$

The latter however is impossible because then there is an $h \in H$ and $v_1 \in V_1$ for which $h(v_1)$ is a nonzero member of W'. For this h, ηh has rank at most $k - i - 1$ on V_1, contrary to assumption that ηH is a k-subspace.

We are now in a position to prove our main result of this section, namely

2.4 Theorem: Let $Y \in \Lambda^{r+1}U$ be not of the form $x \wedge X$ for an $x \in U$ and $X \in \Lambda^r U$. Let $\eta : \Lambda^{r+1}U \to \Lambda^{r+1}U/\langle Y\rangle$ be the natural map. Then ηH_r is an indecomposable $\binom{n-1}{r}$-subspace of $L(\Lambda^r U, \Lambda^{r+1}U/\langle Y\rangle)$. If Y_1 and Y_2 are inequivalent members of $\Lambda^{r+1}U$ not of the form $x \wedge X$ and if η_1 and η_2 are the natural maps from $\Lambda^{r+1}U$ to $\Lambda^{r+1}U/\langle Y_1\rangle$ and $\Lambda^{r+1}U/\langle Y_2\rangle$ respectively then $\eta_1 H_r$ and $\eta_2 H_r$ are inequivalent.

The condition that Y not be of the form $x \wedge X$ means that $Y \notin h(\Lambda^r U)$ for all $h \in H_r$. For Y_1 and Y_2 to be inequivalent means that there is no automorphism A of U for which $Y_1 = A^{(r+1)}(Y_2)$.

By theorems 2.1 and 2.2, ηH_r is indecomposable.

What remains to be proved is that $\eta_1 H_r$ and $\eta_2 H_r$ are inequivalent. Suppose not.

Let α, β, γ be chosen as in section 1 where $V_1 = V_2 = \Lambda^r U$, $W_1 = \Lambda^{r+1}U/\langle Y_1\rangle$, and $W_2 = \Lambda^{r+1}U/\langle Y_2\rangle$. Let $x \in U$ and $h \in \eta_1 H_r$ where h is induced by the exterior multiplication by x. Then $\gamma(h)$ is induced by the exterior multiplication by a unique $x^1 \in U$ and the map $x \to x^1$ is a 1:1 linear map L of U. That L is 1:1 follows from the assumption that Y_1 is not of the form $x \wedge X$ for any $x \in U$ and $X \in \Lambda^r U$.

For any set $x_1, \ldots, x_r$ of independent vectors of U we have

$$0 = \beta(\eta_1(x_o \wedge x_1 \wedge \ldots \wedge x_r))$$

$$= \eta_2(L(x_i) \wedge \alpha(x_1 \wedge \ldots \wedge x_r))$$

and therefore

$$L(x_i)\wedge\alpha(x_1\wedge\ldots\wedge x_r) = 0 .$$

Then

$$\alpha(x_1\wedge\ldots\wedge x_r) = \lambda L(x_1)\wedge\ldots\wedge L(x_r)$$

where λ may depend on the choice of $x_1,\ldots,x_r$. However, linearity and nonsingularity of both α and L implies that λ is constant and we incorporate its rth root in L to obtain

$$\alpha = L^{(r)} .$$

By representing Y_1 as a sum of decomposable (r+1)-vectors it follows that

$$L^{(r+1)}(Y_1) \in \langle Y_2\rangle ,$$

implying that Y_1 and Y_2 are equivalent, contrary to our hypothesis.

When $\binom{n}{r} > n^2$, $\Lambda^r U$ contains infinitely many inequivalent members, because the dimension n^2 of the group acting on $\Lambda^r U$ is less than the dimension $\binom{n}{r}$ of $\Lambda^r U$. In particular, when $n \geq 9$ and $r = 3$ we have $\binom{n}{3} > n^2$. In addition, only finitely many of the equivalence classes can come from trivectors of the form $x \wedge X$ with $x \in U$ and $X \in \Lambda^2 U$ because $\Lambda^2 U$ has only finitely many equivalence classes. Therefore $L(V_1,V_2)$ has infinitely many inequivalent indecomposable k-subspaces when $\dim V_1 = \binom{n}{2}$, $\dim V_2 = \binom{n}{3} - 1$, and $k = \binom{n-1}{2}$.

§3 DECOMPOSABLE SPACES

In this section we show how to construct an infinite number of decomposable k-subspaces of equal dimension in a particular space $L(V,W)$. The construction is best described in terms of matrices and depends on two ways of generating spaces of matrices from a given space.

Suppose X is a space of $m\times n$ matrices with a basis $x_1, x_2, \ldots, x_d$. Let $y_1, y_2, \ldots, y_n$ be $m\times d$ matrices defined so that the jth column of y_i is equal to the ith column of x_j. When the matrices $y_1, \ldots, y_n$ are linearly independent we say that X is <u>column full</u> in that there is no column operation which simultaneously reduces a fixed one of the columns of each x_i to zero. If X is column full we define $f(X) = \langle y_1, \ldots, y_n \rangle$, an n-dimensional space of $m\times d$ matrices. Of course $f(X)$ depends upon the given basis of X but our notation deliberately suppresses this since a different choice of basis will give a space equivalent to $f(X)$ under a column operation. Moreover, from the definition it follows that, for any $m\times m$ non-singular p and $n\times n$ non-singular q, $f(pX) = pf(X)$ and $f(Xq) = f(X)$. Thus f is a well-defined 1:1 operation on equivalence classes, clearly involutory, which maps column full d-dimensional spaces of $m\times n$ matrices into column full n-dimensional spaces of $m\times d$ matrices.

3.1. <u>Lemma</u>: <u>Let X be a column full space of</u> $m\times n$ <u>matrices</u>. <u>Then</u> X <u>contains a non-zero matrix of rank less than</u> m <u>if and only if the equivalence class of</u> $f(X)$ <u>contains a space whose matrices all have their</u> (1,1) <u>entries equal to zero</u>.

Proof: Suppose that X contains a non-zero matrix of rank less than m. Since f respects equivalence classes we may replace X by a space containing a matrix with zero first row and take this matrix as the first member of a basis of X. Then, by definition of f, $f(X)$ has a basis whose (1,1) positions are all zero. The converse follows by reversing the argument.

We turn now to another correspondence on equivalence classes. For any space X of $m\times n$ matrices let

$$g(X) = \{y \in M_{nm} : \text{trace}\,(xy) = 0 \text{ for all } x \in X\}$$

If p and q are non-singular $m\times m$ and $n\times n$ matrices then

$$g(pXq) = q^{-1}g(X)p^{-1}.$$

Thus g, like f, respects equivalence classes, is also involutory, and maps d-dimensional spaces of $m\times n$ matrices into $(mn-d)$-dimensional spaces of $n\times m$ matrices.

3.2. Lemma: *Let X be any space of $m\times n$ matrices. Then some space equivalent to X has all its (1,1) entries equal to zero if and only if $g(X)$ contains a rank 1 matrix.*

Proof: From the definition of g we have that $g(X)$ contains the matrix E_{11} (with a one in the (1,1) position and zeros elsewhere) if and only if the matrices of X all have zero (1,1) position. The lemma now follows since g preserves equivalence classes.

Putting these two lemmas together gives

3.3. Lemma: Suppose that $m \le n$ and X is a column full d-dimensional space of $m \times n$ matrices. Then X is an m-space if and only if $g(f(X))$ (which is an $(md-n)$-dimensional space of $d \times m$ matrices) contains no rank 1 matrix.

Notice that every $(md-n)$-dimensional space Y of $d \times m$ matrices containing no rank 1 matrix arises in the form $g(f(X))$ for some column full space X. This is because $Z = g(Y) = g^{-1}(Y)$ is not equivalent to a space with all (1,1) entries equal to zero, hence is column full, and so of the form $Z = f^{-1}(X) = f(X)$.

3.4. Theorem: If $n \ge 4$ there are an infinite number of inequivalent n-dimensional n-spaces of $n \times (n^2 - 2)$ matrices.

Proof: By Lemma 3.3 and the remark following it we need only show that there are an infinite number of inequivalent 2-dimensional spaces of $n \times n$ matrices which do not contain a rank 1 matrix.

Let $G = PGL(2,F)$ be the group of all linear fractional mappings $z \to \frac{pz+q}{rs+s}$, $ps - rq \ne 0$, of the complex projective line $\pi = F \cup \{\infty\}$. G permutes $\pi(n)$, the set of n-subsets of π, and in this action it has an infinite number of orbits (for each n-set T there are only a finite number of cross-ratios formed by 4 points selected from T, G preserves cross-ratios, and so for each orbit Γ in $\pi(n)$ there are only a finite number of cross ratios formed by 4 points taken from an n-set in Γ). We shall show that, for every orbit of G in $\pi(n)$, there is an equivalence class of 2-dimensional spaces of $n\times n$ matrices which do not contain a rank 1 matrix, and that distinct orbits give rise to distinct equivalence classes.

So let Γ be any orbit of G in $\pi(n)$ and let $T = \{\rho_1, \rho_2, \ldots, \rho_n\}$ be one of the n-sets of Γ. Represent each ρ_i as σ_i/τ_i with $\sigma_i, \tau_i \in F$ and consider the space X generated by

$$a = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \quad \text{and} \quad b = \operatorname{diag}(\tau_1, \ldots, \tau_n) .$$

Notice that X contains no rank 1 matrix because the ρ_i are all distinct. The equivalence class of X depends neither on the particular representation of ρ_i as σ_i/τ_i nor on the ordering $\rho_1, \rho_2, \ldots, \rho_n$; it is therefore determined by T alone and we may denote it by $E(T)$.

Suppose that T' is any other n-set in the orbit Γ. Then there is a transformation $z \to \frac{\alpha z+\beta}{\gamma z+\delta}$ of G which maps T to T' and we may let $T' = \{\rho_1', \ldots, \rho_n'\}$ where $\rho_i' = \frac{\alpha\rho_i+\beta}{\gamma\rho_i+\delta}$. The space X is generated also by

$$\alpha a + \beta b = \operatorname{diag}(\alpha\sigma_1 + \beta\tau_1, \ldots, \alpha\sigma_n + \beta\tau_n) \quad \text{and}$$

$$\gamma a + \delta b = \operatorname{diag}(\gamma\sigma_1 + \delta\tau_1, \ldots, \gamma\sigma_n + \delta\tau_n)$$

and, since $\rho_i' = \frac{\alpha\sigma_i+\beta\tau_i}{\gamma\sigma_i+\delta\tau_i}$, we must have $E(T) = E(T')$; in other words, the equivalence class of X is determined by Γ alone.

Finally, to show that different orbits of G in $\pi(n)$ produce different equivalence classes we consider an arbitrary space Y in $E(T)$, take any basis c , d for Y , and show that the homogeneous polynomial in x and y $\det(xc - yd)$ has the factorization $\prod_{i=1}^{n} (x\varepsilon_i - y\eta_i)$ where $\{\varepsilon_1/\eta_1, \ldots, \varepsilon_n/\eta_n\}$ is one of the n-sets of Γ . Since Y is equivalent to X there exist non-singular matrices p , q and constants $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$ such that

$$pcq = \alpha a + \beta b , \quad pdq = \gamma a + \delta b .$$

However,

$$\alpha a + \beta b = \operatorname{diag}(\sigma_1', \ldots, \sigma_n') , \quad \gamma a + \delta b = \operatorname{diag}(\tau_1', \ldots, \tau_n')$$

where $\sigma_i'/\tau_i' = \dfrac{\alpha\sigma_i + \beta\tau_i}{\gamma\sigma_i + \delta\tau_i} = \dfrac{\alpha\rho_i + \beta}{\gamma\rho_i + \delta}$ and

$$\det(xc - yd) = \det(x(\alpha a + \beta b) - y(\gamma a + \delta b))/\det(pq)$$

$$= \left(\prod_{i=1}^{n} (x\sigma_i' - y\tau_i')\right)/\det(pq)$$

$$= \left(\prod_{i=1}^{n} (x\varepsilon_i - y\eta_i)\right)$$

where the set $\{\varepsilon_1/\eta_1, \ldots, \varepsilon_n/\eta_n\} = \{\sigma_1'/\tau_1', \ldots, \sigma_n'/\tau_n'\} \in \Gamma$.

1. L.B. Beasley, Spaces of matrices of equal rank, Linear Albegra and Appl. 38 (1981), 227-237.

2. R. Westwick, Spaces of linear transformation of equal rank, Linear Algebra and Appl. 5 (1972), 49-64.