LARGE SPACES OF MATRICES OF BOUNDED RANK

By M. D. ATKINSON *and* S. LLOYD

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IN THIS paper we consider subspaces $\mathscr{X}$ of $\mathscr{M}_{mn}$, the space of all $m \times n$ matrices with entries in some given field, with the property that each matrix of $\mathscr{X}$ has rank at most r. In [**2**] Flanders showed that such spaces necessarily have dimension at most $\max(mr, nr)$ and he determined the spaces of precisely this dimension. We shall extend this work by classifying the spaces of dimension slightly lower than this upper bound. Our results depend on the (often unstated) assumption that the ground field has at least $r+1$ elements but, unlike Flanders, we do not need to exclude the characteristic 2 case.

If every matrix in the space $\mathscr{X}$ has rank at most r the same is clearly true of the space $P\mathscr{X}Q = \{PXQ : X \in \mathscr{X}\}$ where P, Q are non-singular $m \times m$, $n \times n$ matrices respectively. This *equivalent* space $P\mathscr{X}Q$ can also be derived from $\mathscr{X}$ by performing row and column operations to all matrices of $\mathscr{X}$ simultaneously. A wide class of examples is provided by spaces equivalent to subspaces of the space $\mathscr{R}(p, q)$ of all matrices of the form $\begin{pmatrix} A & B \\ C & O \end{pmatrix}$ where A is a $p \times q$ matrix and $p+q=r$. These examples we shall call *r-decomposable* and they clearly consist of matrices of rank at most r. It was shown in [**1**] that any 2-dimensional space of matrices of rank at most r is r-decomposable.

We shall find it useful to exploit the duality between $\mathscr{M}_{mn}$ and $\mathscr{M}_{nm}$ defined by the bilinear form

$$\tau(X, Y) = \operatorname{trace}(XY) \quad \text{for } X \in \mathscr{M}_{mn},\ Y \in \mathscr{M}_{nm}.$$

If $\mathscr{X}$ is a subspace of $\mathscr{M}_{mn}$ of dimension d we shall denote by $\mathscr{X}^*$ its annihilator in $\mathscr{M}_{nm}$ with respect to τ; $\mathscr{X}^*$ has codimension d in $\mathscr{M}_{nm}$. A simple calculation shows that $(P\mathscr{X}Q)^* = Q^{-1}\mathscr{X}^*P^{-1}$ and hence equivalent spaces have equivalent annihilators.

THEOREM. *Let $\mathscr{X}$ be a space of $m \times n$ matrices over some field with at least $r+1$ elements. Suppose that every matrix in $\mathscr{X}$ has rank at most r and that* $\dim \mathscr{X} \geq \max(mr, nr) - r + 1$. *Then, if $m < n$, $\mathscr{X}$ is equivalent to a subspace of $\mathscr{R}(r, 0)$ (matrices whose last $m-r$ rows are zero) while, if $m > n$, $\mathscr{X}$ is equivalent to a subspace of $\mathscr{R}(0, r)$. If $m = n$ one of these two possibilities must occur except if* $\dim \mathscr{X} = nr - r + 1$ *when the only further possibilities are that $\mathscr{X}$ is equivalent to $\mathscr{R}(r-1, 1)$ or to $\mathscr{R}(1, r-1)$.*

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Much of the proof is concerned with the study of the rank 1 matrices in the space. In particular it is convenient to denote by F_i the rank 1 matrix which consists of zeros except for a 1 in the (i, i) position. A set of rank 1 matrices $M_1, M_2, \ldots, M_k$ is said to be an *orthogonal* set of rank 1 matrices if some linear combination of $M_1, \ldots, M_k$ has rank k. Clearly $F_1, \ldots, F_k$ is an orthogonal set. Conversely we have

LEMMA 1. *Every orthogonal set of k rank 1 matrices is (simultaneously) equivalent to $F_1, \ldots, F_k$.*

Proof. The case $k=1$ is well known and provides the base for an induction on k. Assume that $k>1$ and that $M_1, \ldots, M_k$ is an orthogonal set of rank 1 matrices. It is clear that $M_1, \ldots, M_{k-1}$ is an orthogonal set and so we may assume that row and column operations have already been used to transform $M_1, \ldots, M_{k-1}$ into $F_1, \ldots, F_{k-1}$. Let $M_k = \begin{pmatrix} U & V \\ W & X \end{pmatrix}$ where U is a $(k-1)\times(k-1)$ matrix. If $X=0$ then, since M_k has rank 1, one of V and W is 0; but then all linear combinations of $M_1, \ldots, M_k$ have rank less than k. Hence X contains some non-zero entry and by row and column operations which leave $F_1, \ldots, F_{k-1}$ unchanged we may transform M_k so that it has a 1 in the (k, k) position and zeros elsewhere in the kth row and column. Then, since M_k has rank 1, we must have $M_k = F_k$ as required.

LEMMA 2. *If A is a matrix of the form $\begin{pmatrix} \alpha & u \\ v & B \end{pmatrix}$ where u is a row vector, v is a column vector and B is a matrix of rank $k-1$, then, for some scalar x, rank $(xF_1 + A) = k$.*

Proof. B contains a $(k-1)\times(k-1)$ non-singular submatrix B_1 say. The determinant of the $k\times k$ submatrix $\begin{pmatrix} \alpha+x & u_1 \\ v_1 & B_1 \end{pmatrix}$ of $xF_1 + A$, where u_1 consists of the elements of u in the columns of B_1 and v_1 is defined similarly, is a polynomial of degree 1 in x and the coefficient of x is $\det B_1$. Hence, for some value of x, $xF_1 + A$ has a $k\times k$ non-singular submatrix and thus has rank k.

LEMMA 3. *Let $\mathscr{X}$ be a space of matrices generated by rank 1 matrices and containing a matrix of rank k. Then $\mathscr{X}$ contains k orthogonal rank 1 matrices.*

Proof. Let $M_1, \ldots, M_t$ be a basis of rank 1 matrices for $\mathscr{X}$. We shall prove by induction on k that such a basis contains a set of k orthogonal rank 1 matrices. This result is obvious if $k=1$ so we now assume that $k>1$ and that the statement has been proved for smaller values than k.

By assumption $\mathscr{X}$ contains a matrix of rank k and by passing to an equivalent space we may take this to be $A = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ where I_k is the $k \times k$ identity matrix. Since $a_{11} \neq 0$ one of the basis matrices, M_1 say, has non-zero (1, 1) position. By subtracting multiples of the first row and column from the other rows and columns we may take M_1 to be F_1; in so doing we change the first row and column of A so that it now has the form

$$\begin{pmatrix} \alpha & u_1 & u_2 \\ v_1 & I_{k-1} & 0 \\ v_2 & 0 & 0 \end{pmatrix}$$

For any matrix $X \in \mathscr{X}$ let X' be the matrix obtained by deleting the first row and first column, and let $\mathscr{X}'$ be the space of all such X'. This space $\mathscr{X}'$ has a basis of rank 1 matrices (viz. a subset of $M_2', \ldots, M_t'$, since $M_1' = 0$) and contains a matrix of rank $k-1$. Applying the inductive hypothesis to $\mathscr{X}'$ we obtain $k-1$ orthogonal rank 1 matrices among $M_2', \ldots, M_t'$ which, by renumbering, we may take to be $M_2', \ldots, M_k'$. It follows that some linear combination C of $M_2, \ldots, M_k$ has C' of rank $k-1$. But then Lemma 2 shows that some linear combination of M_1 and C has rank k; therefore $M_1, \ldots, M_k$ is an orthogonal set as required.

LEMMA 4. *Let $\mathscr{X}$ be a space of matrices all having rank at most r. Then either of the following is a sufficient condition for $\mathscr{X}$ to be r-decomposable:*
(i) *$\mathscr{X}$ contains r orthogonal rank 1 matrices,*
(ii) *$\mathscr{X}$ is generated by rank 1 matrices.*

Proof. We first prove, by induction on r, that condition (i) implies the r-decomposability of $\mathscr{X}$. If $r = 0$ this is trivially true so we assume that $r > 0$ and that the result is true for smaller values. By Lemma 1 we may assume that $F_1, \ldots, F_r$ belong to $\mathscr{X}$. As in the proof of Lemma 3, for each matrix $X \in \mathscr{X}$ we let X' be the matrix obtained by deleting the first row and column of X. Then, by Lemma 2, the space $\mathscr{X}'$ of matrices X' consists of matrices of rank at most $r-1$. Since also $\mathscr{X}'$ contains $r-1$ orthogonal rank 1 matrices (viz. $F_2', \ldots, F_r'$) we may apply the inductive hypothesis to $\mathscr{X}'$. Hence, for some p, q with $p+q = r-1$, $\mathscr{X}'$ is equivalent to a subspace of $\mathscr{R}(p, q)$. The row and column operations which demonstrate this equivalence may clearly be induced by operations on the rows and columns of the matrices in $\mathscr{X}$ and therefore we may take the matrices of $\mathscr{X}$ to have the form $\begin{pmatrix} T & U \\ V & 0 \end{pmatrix}$ where T is a $(p+1) \times (q+1)$ matrix.

Let $\mathscr{U}$, $\mathscr{V}$ be respectively the spaces consisting of all the matrices U, all the matrices V. Also let x, y be respectively the maximal rank of a matrix in $\mathcal{U}$, $\mathcal{V}$. Since each $U \in \mathcal{U}$ has at most $p+1$ rows we have $x \leq p+1$. Similarly $y \leq q+1$ and consequently $x+y \leq p+1+q+1=r+1$. However we shall show that $x+y \leq r$. By definition of x and y there exist matrices $A_1, A_2 \in \mathcal{X}$ of the form $A_1 = \begin{pmatrix} T_1 & U_1 \\ V_1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} T_2 & U_2 \\ V_2 & 0 \end{pmatrix}$ with rank $(U_1) = x$, rank $(V_2) = y$. Then U_1 has an $x \times x$ non-singular submatrix $\bar{U}_1$ and we let $\bar{U}_2$ be the corresponding submatrix of U_2. Similarly V_2 has a $y \times y$ non-singular submatrix $\bar{V}_2$ and we let $\bar{V}_1$ be the corresponding submatrix of V_1. The equation $\det(\alpha \bar{U}_1 + \beta \bar{U}_2) = 0$ has at most x solutions for the ratio α/β while the equation $\det(\alpha \bar{V}_1 + \beta \bar{V}_2) = 0$ has at most y solutions for the ratio β/α. Since the field has at least $r+1$ elements there are, including infinity, at least $r+2$ ratios. Hence for some α, β both $\alpha \bar{U}_1 + \beta \bar{U}_2$ and $\alpha \bar{V}_1 + \beta \bar{V}_2$ are non-singular. It follows that $\alpha U_1 + \beta U_2$ has rank x, $\alpha V_1 + \beta V_2$ has rank y and so $\alpha A_1 + \beta A_2$ has rank at least $x+y$. But $\alpha A_1 + \beta A_2 \in \mathcal{X}$ and therefore has rank at most r, thus $x+y \leq r$ as required.

We now recall the matrices $F_2, \ldots, F_r$ and observe that as a result of the transformations applied to $\mathcal{X}$ they have become matrices $G_2, \ldots, G_r$ of the form $G_i = \begin{pmatrix} H_i & K_i \\ L_i & 0 \end{pmatrix}$ with zero first row and column. Clearly rank $((H_2 K_2) + \cdots + (H_r K_r)) \leq p$ and rank $(L_2 + \cdots + L_r) \leq q$, but since

$$\begin{aligned} r-1 &= \operatorname{rank}(F_2 + \cdots + F_r) = \operatorname{rank}(G_2 + \cdots + G_r) \\ &\leq \operatorname{rank}((H_2 K_2) + \cdots + (H_r K_r)) + \operatorname{rank}(L_2 + \cdots + L_r) \\ &\leq p+q = r-1 \end{aligned}$$

we must have rank $(L_2 + \cdots + L_r) = q$.

Applying Lemma 3 to the subspace $\langle L_2, \ldots, L_r \rangle$ of $\mathcal{V}$ we deduce that $\mathcal{V}$ contains q orthogonal rank 1 matrices; similarly $\mathcal{U}$ contains p orthogonal rank 1 matrices.

We complete the proof of the first part of the lemma by another application of the inductive hypothesis. Since $x+y \leq r$ we must have $x \leq p$ or $y \leq q$. Suppose that $y \leq q$ (the case $x \leq p$ is similar). Then, as $q < r$, $\mathcal{V}$ is q-decomposable and there exist row and column operations which transform $\mathcal{V}$ into a subspace of $\mathcal{R}(s, t)$ for some s, t with $s+t=q$. These row and column operations clearly may be induced by row and column operations on the matrices of $\mathcal{X}$. Thus the matrices of $\mathcal{X}$ are equivalent to matrices of the form

$$\left(\begin{array}{cc|c} \multicolumn{2}{c}{T} & U \\ \hline X & Y & \\ Z & 0 & 0 \end{array} \right)$$

with T a $(p+1)\times(q+1)$ matrix and X an $s\times t$ matrix. But such matrices are contained in $\mathcal{R}(p+1+s, t)$ and since $p+1+s+t=p+q+1=r$ this shows that $\mathcal{X}$ is r-decomposable.

To prove the second part we suppose that $\mathcal{X}$ is generated by rank 1 matrices and let t be the maximum rank of a matrix in $\mathcal{X}$. By Lemma 3 $\mathcal{X}$ contains t orthogonal rank 1 matrices and hence, by the first part, is t-decomposable. But $t\leqslant r$ and hence $\mathcal{X}$ is r-decomposable.

LEMMA 5. *Let $\mathcal{X}$ be a space of $n\times n$ matrices all of which have rank at most r. Suppose that* $\dim\mathcal{X}=nr-k$ *and that $\mathcal{X}$ contains the matrix* $\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$. *Then $\mathcal{X}$ has a subspace of dimension at least r^2-k whose matrices all have the form* $\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}$ *where A is an $r\times r$ matrix.*

Proof. Throughout the proof all matrices will be partitioned as $\begin{pmatrix} A & B\\ C & D\end{pmatrix}$ where A is an $r\times r$ matrix. By Lemma 1 of [2] for all such matrices we have $D=0$ and $CB=0$. We define several spaces related to $\mathcal{X}$:

$$\mathcal{F}=\left\{\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}:\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\in\mathcal{X}\quad\text{for some } C\right\},$$

$$\mathcal{A}=\left\{\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\in\mathcal{X}\quad\text{for some } B, C\right\},$$

$$\mathcal{B}=\left\{\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix}:\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\in\mathcal{X}\quad\text{for some } A, C\right\},$$

$$\mathcal{C}_0=\left\{\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\in\mathcal{X}\right\}.$$

Clearly $\mathcal{F}\subseteq\mathcal{A}\oplus\mathcal{B}$ and so $x=\dim\mathcal{A}+\dim\mathcal{B}-\dim\mathcal{F}\geqslant 0$. Moreover $\dim\mathcal{A}\leqslant r^2$ and so $y=r^2-\dim\mathcal{A}\geqslant 0$. For $\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\in\mathcal{X}$ the mapping $\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\mapsto\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}$ is onto $\mathcal{F}$ and has kernel $\mathcal{C}_0$; hence $\dim\mathcal{F}+\dim\mathcal{C}_0=\dim\mathcal{X}$.

Let $\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix}\in\mathcal{B}$ so that $\begin{pmatrix} A & B\\ C & 0\end{pmatrix}\in\mathcal{X}$ for some A, C, and let $\begin{pmatrix} 0 & 0\\ C_0 & 0\end{pmatrix}\in\mathcal{C}_0$. Then $\operatorname{trace}(CB)=0$ and, since $\begin{pmatrix} A & B\\ C+C_0 & 0\end{pmatrix}\in\mathcal{X}$, $\operatorname{trace}(C+C_0)B=0$; hence $\operatorname{trace}(C_0B)=0$. We may identify $\mathcal{B}$, $\mathcal{C}_0$ with spaces of $r\times(n-r)$,

$(n-r)\times r$ matrices respectively and, with this identification, each annihilates the other with respect to the trace bilinear form. Thus $\dim \mathscr{B} + \dim \mathscr{C}_0 \leq r(n-r)$ and so $z = r(n-r) - \dim \mathscr{B} - \dim C_0 \geq 0$.

We then have

$$\begin{aligned}\dim \mathscr{X} = \dim \mathscr{F} + \dim \mathscr{C}_0 &= -x + \dim \mathscr{A} + \dim \mathscr{B} + \dim \mathscr{C}_0\\ &= -x + r^2 - y + \dim \mathscr{B} + \dim \mathscr{C}_0\\ &= -x + r^2 - y + r(n-r) - z\\ &= nr - (x+y+z)\end{aligned}$$

and it follows that $x+y+z=k$.

Let $\mathscr{G}_1 = \left\{\begin{pmatrix} A & 0\\ C & 0\end{pmatrix} \in \mathscr{X}\right\}$. Then for any $\begin{pmatrix} A & 0\\ C & 0\end{pmatrix} \in \mathscr{G}_1$ the argument used above to show that $\mathscr{C}_0$ annihilates $\mathscr{B}$ shows also that the matrix C annihilates $\mathscr{B}$. By definition of z, $\mathscr{C}_0$ has codimension z in the annihilator of $\mathscr{B}$ and consequently the subspace $\mathscr{G}_0$ of $\mathscr{G}_1$ consisting of all those $\begin{pmatrix} A & 0\\ C & 0\end{pmatrix} \in \mathscr{X}$ for which $\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix} \in \mathscr{C}_0$ has codimension at most z in $\mathscr{G}_1$. The map $\begin{pmatrix} A & 0\\ C & 0\end{pmatrix} \mapsto \begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}$ maps $\mathscr{G}_1$ onto $\mathscr{F} \cap \mathscr{A}$ and hence the image of $\mathscr{G}_0$ under this map has codimension at most z in $\mathscr{F} \cap \mathscr{A}$. However this image, $\mathscr{X}_0$, consists of all matrices $\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}$ for which $\begin{pmatrix} A & 0\\ C & 0\end{pmatrix} \in \mathscr{X}$ for some $\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix} \in \mathscr{C}_0$ i.e. $\mathscr{X}_0$ consists of matrices $\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix} \in \mathscr{X}$. We now deduce

$$\begin{aligned}\dim \mathscr{X}_0 &\geq \dim \mathscr{F} \cap \mathscr{A} - z\\ &= \dim \mathscr{F} + \dim \mathscr{A} - \dim(\mathscr{F} + \mathscr{A}) - z\\ &\geq \dim \mathscr{F} + \dim \mathscr{A} - \dim(\mathscr{A} \oplus \mathscr{B}) - z \quad \text{since } \mathscr{F} + \mathscr{A} \leq \mathscr{A} \oplus \mathscr{B}\\ &= \dim \mathscr{F} - \dim \mathscr{B} - z\\ &= \dim \mathscr{A} - x - z\\ &= r^2 - x - y - z\\ &= r^2 - k\end{aligned}$$

and this completes the proof.

LEMMA 6. *Let $\mathscr{X}$ be a space of $r\times r$ matrices of dimension at least $r^2 - r + 1$ defined over an algebraically closed field. Then $\mathscr{X}$ contains r orthogonal rank 1 matrices.*

Proof. Let $\mathscr{Y}$ be the subspace generated by the rank 1 matrices in $\mathscr{X}$ and let t be the maximum rank of any matrix in $\mathscr{Y}$. We shall prove that

$t=r$ from which the conclusion will follow by applying Lemma 3 to $\mathcal{Y}$. By Lemma 4 $\mathcal{Y}$ is t-decomposable and we may replace $\mathcal{X}$ by an equivalent space so that every matrix of $\mathcal{Y}$ has the form $\begin{pmatrix} U & V \\ W & 0 \end{pmatrix}$ where U is a $p\times q$ matrix with $p+q=t$. If $p=r$ or $q=r$ then clearly $t=r$ so from now on assume that $r-p$, $r-q$ are positive. Let

$$\mathcal{Z}=\left\{\begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}\in\mathcal{X},\quad Z \text{ an } (r-p)\times(r-q) \text{ matrix}\right\}.$$

The condition on $\dim\mathcal{X}$ implies that $\dim\mathcal{Z}\geq(r-p)(r-q)-r+1$ and, by definition of $\mathcal{Y}$, $\mathcal{Z}$ contains no rank 1 matrix. However the dimension of the (irreducible) variety of $(r-p)\times(r-q)$ matrices of rank at most 1 is $(r-p)+(r-q)-1=2r-t-1$ (Theorem 2.1 of [3]). Since this variety intersects $\mathcal{Z}$ only in the zero matrix the dimension theorem of algebraic geometry yields

$$2r-t-1+\dim\mathcal{Z}-(r-p)(r-q)\leq 0.$$

Hence $2r-t-1+(r-p)(r-q)-r+1-(r-p)(r-q)\leq 0$ from which we obtain $t\geq r$ as required.

This lemma has an equivalent formulation which we feel is interesting enough to be mentioned even though we make no use of it. Consider any $r-1$ $r\times r$ matrices and let $\mathcal{A}$ be the space they generate. The annihilator space with respect to the trace bilinear form has dimension at least r^2-r+1 and so contains r orthogonal rank 1 matrices. Thus there is, by Lemma 1, a space equivalent to $\mathcal{A}$ every matrix A of which satisfies

$$\operatorname{trace}(AF_i)=0, \qquad i=1,2,\ldots,r.$$

But this condition is just that A has zero main diagonal. This gives

COROLLARY. *Given any $r-1$ $r\times r$ matrices with entries in an algebraically closed field there exist row and column operations which simultaneously reduce all their main diagonals to zero.*

This lemma and corollary are not true for an arbitrary field. For suppose that $r=3$ and consider the 7-dimensional space of all matrices of the form

$$\begin{pmatrix} a & b & c \\ d & f & g \\ e & -g & f \end{pmatrix}$$

Over the reals such a matrix has rank 1 only if $f=g=0$. However the 5-dimensional space of all $\begin{pmatrix} a & b & c \\ d & 0 & 0 \\ e & 0 & 0 \end{pmatrix}$ cannot contain 3 orthogonal rank 1 matrices because all matrices in this space have rank at most 2. The annihilator of the 7-dimensional space is spanned by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and hence, over the reals, these matrices are not equivalent to matrices with zero diagonal.

LEMMA 7. *Let $\mathscr{X}$ be a space of matrices all of rank at most r and containing a matrix of rank r. Then $\mathscr{X}$ is generated by matrices of rank r.*

Proof. Let $A \in \mathscr{X}$ and have rank r and let A' be an $r \times r$ non-singular submatrix of A. Let B be any other matrix in $\mathscr{X}$ and let B' be the corresponding $r \times r$ submatrix. The equation $\det(\lambda A' + B') = 0$ has at most r solutions and since the ground field has $r+1$ elements $\lambda A' + B'$ is non-singular for some λ. But then B is a linear combination of A and $\lambda A + B$ both of which have rank r.

PROOF OF THEOREM. Let F be the field over which the matrices of $\mathscr{X}$ are defined, let $\bar{F}$ be its algebraic closure and let $\bar{\mathscr{X}}$ be the vector space over $\bar{F}$ of all $\bar{F}$-linear combinations of matrices in $\mathscr{X}$. Choose an F-basis $A_1, A_2, \ldots$ for $\mathscr{X}$; this set is then also an $\bar{F}$-basis for $\bar{\mathscr{X}}$. Consider any set of $r+1$ rows and $r+1$ columns and for every matrix A let A' be the $(r+1) \times (r+1)$ submatrix of A consisting of these rows and columns. The polynomial $\det(\sum x_i A_i')$ is then zero for all values in F of the variables $x_1, x_2, \ldots$. This polynomial is homogeneous of total degree at most $r+1$ and no terms x_i^{r+1} occur since the coefficient of such a term is $\det A_i'$ which is zero. Hence the polynomial is of degree at most r in each variable and since $|F| \geq r+1$ it vanishes identically. Thus every $(r+1) \times (r+1)$ submatrix of each matrix of $\bar{\mathscr{X}}$ is singular and so $\bar{\mathscr{X}}$ also consists of matrices of rank at most r.

We now use the lemmas to prove that $\bar{\mathscr{X}}$ is r-decomposable. Let $\bar{\mathscr{X}}_1$ be the space of matrices obtained by adding $m-n$ columns of zeros to the matrices of $\bar{\mathscr{X}}$ if $m \geq n$, or by adding $n-m$ rows of zeros if $n \geq m$. According to Flanders theorem [**2**] $\bar{\mathscr{X}}_1$ (indeed $\mathscr{X}$) must contain a matrix of rank r precisely and by passing to an equivalent space this can be taken as $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Then Lemma 5 shows that $\bar{\mathscr{X}}_1$ contains a subspace of dimension at least $r^2 - r + 1$ whose matrices have the form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ with A an $r \times r$ matrix. By Lemma 6 this subspace contains r orthogonal rank 1 matrices and hence, by Lemma 4, $\bar{\mathscr{X}}_1$ is r-decomposable.

We interpret each matrix in $\bar{\mathscr{X}}$ as a linear mapping from the space V_m of row m-vectors over $\bar{F}$ to the space V_n of row n-vectors over $\bar{F}$. Then each matrix in $\bar{\mathscr{X}}_1$ can be interpreted as a linear mapping from $V_m \oplus V_{n-m}$ to V_n, where the matrix acts on V_m as a matrix in $\bar{\mathscr{X}}$ and maps V_{n-m} to zero. The condition that $\bar{\mathscr{X}}_1$ is r-decomposable is that $V_m \oplus V_{n-m}$ has a subspace U_1 of codimension p which is mapped by every matrix in $\bar{\mathscr{X}}_1$ into a subspace W of V_n of dimension q. But $U = U_1 \cap V_m$ has codimension $\leq p$ in V_m and is mapped by all matrices in $\bar{\mathscr{X}}_1$ into W. Since $U \leq V_m$ the matrices of $\bar{\mathscr{X}}$ map U into W and hence $\bar{\mathscr{X}}$ itself is r-decomposable.

Some space equivalent to $\bar{\mathscr{X}}$ is therefore contained in $\mathscr{R}(p, q)$ for some p, q with $p+q=r$. One easily calculates that $\dim \mathscr{R}(p, q) = np + mq - pq$ and if we put $\dim \mathscr{X} = \max(mr, nr) - k$ with $k \leq r-1$ we have

$$\max(mr, nr) - k \leq np + mq - pq.$$

If $m < n$ this leads to

$$nr - k \leq np + mq - pq \leq np + (n-1)q - pq = nr - q - pq$$

from which

$$pq + q \leq k \leq r - 1 = p + q - 1$$

and so $q = 0$. Similarly if $m > n$ we have $p = 0$. If $m = n$ then

$$nr - k \leq n(p+q) - pq - nr - pq$$

and so $pq \leq k \leq p + q - 1$ and therefore $(p-1)(q-1) \leq 0$. Hence if p and q are both non-zero we have $k = r - 1$ and one of p and q is 1.

The theorem therefore holds in the algebraically closed field $\bar{F}$ and we now deduce it for the original field F. To show that the r-decomposability is realisable over F we must show that U and W above have bases of vectors with entries in F.

Observe that, for any matrix $X \in \mathscr{X}$ of exact rank r, we have $\ker X \leq U$. If $q = 0$ such a matrix has null space precisely U. But the null space of a matrix over F is spanned by F-vectors and so in this case U, and certainly W, is spanned by F-vectors.

If $q = 1$ every matrix of rank r in $\mathscr{X}$ has a null space of codimension 1 in U. If these null spaces are not all equal then U will be generated by these null spaces and again will have a basis of F-vectors. Since W is spanned by the image of a matrix in $\mathscr{X}$ of rank r, W also has a basis of one F-vector. If the null spaces of the matrices of rank r in $\mathscr{X}$ are all equal then by Lemma 7 all matrices of $\mathscr{X}$ map some $(m-r)$-dimensional subspace to 0 and hence $\mathscr{X}$ is equivalent over F to a subspace of $\mathscr{R}(r, 0)$.

The cases $q = 0$, $p = 1$ can be treated in a similar way by regarding $\bar{\mathscr{X}}$ as a space of mappings from a vector space of column n-vectors to a space of column m-vectors.

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Department of Computing Mathematics
University College
Cardiff