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Extensions to the Kronecker–Weierstrass Theory of Pencils

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The Kronecker–Weierstrass theory of pencils is extended to give a necessary and sufficient condition that two $2 \times m \times n$ tensors are equivalent. The connection between equivalence class representatives and the triple transitivity of PGL$(2,F)$ is discussed. One consequence of the discussion is that the number of inequivalent $2 \times 3 \times n$ tensors is finite. An efficient algorithm is given for testing the condition which ultimately depends on a fast pattern matching algorithm.

1. INTRODUCTION

Let U,V,W be finite dimensional vector spaces over a field F and let ϕ, θ be two trilinear forms from $U \times V \times W$ into F. We say that ϕ, θ are *equivalent* if there exist automorphisms α,β,γ of U,V,W such that

$$\phi(u,v,w) = \theta(\alpha(u),\beta(v),\gamma(w)) \qquad \text{for all} \quad (u,v,w) \in U \times V \times W$$

A trilinear form can be defined by its effect $a_{ijk} = \phi(e_i,f_j,g_k)$ on the triples (e_i,f_j,g_k) drawn from fixed bases of U,V,W. Then the definition of equivalence becomes the condition that two third order tensors (a_{ijk}), (b_{ijk}) satisfy

$$a_{ijk} = \sum_{r,s,t} u_{ir}v_{js}w_{kt}b_{rst} \qquad \text{for some non-singular matrices } (u_{ir}),(v_{js}),(w_{kt})$$

A trilinear form $\phi : U \times V \times W \to F$ gives rise, in a natural way, to a bilinear mapping $\zeta : U \times V \to W^*$, the dual of W, through the equation

$$\zeta(u,v)(w) = \phi(u,v,w)$$

so the tensor equivalence equations also describe the natural equivalence on bilinear mappings.

In this paper we shall be considering the case that one of U,V,W (W say) has dimension 2 (dimension 0 is trivial and dimension 1 is easy). In this case, putting $G_k = (a_{ijk})$, $H_k = (b_{ijk})$, $k = 1,2$, the tensor equivalence condition can be formulated so that it is the existence of non-singular matrices P,Q,X such that

$$PG_1Q = x_{11}H_1 + x_{12}H_2$$
$$PG_2Q = x_{21}H_1 + x_{22}H_2$$

where, if G_k, H_k are $m \times n$ matrices, P, Q and X are $m \times m$, $n \times n$ and 2×2 respectively.

There is an old theory dating back to Kronecker and Weierstrass which shows that every pair of $m \times n$ matrices can be transformed by pre-multiplication and post-multiplication by two non-singular matrices to a canonical pair whose entries are largely zeros and ones. A good account of this theory is given in [2]. Kronecker–Weierstrass equivalence of pairs (G_1, G_2) and (H_1, H_2) of $m \times n$ matrices (the equations above with $X = I$) is a stronger equivalence than tensor equivalence. Therefore there is the possibility that distinct Kronecker–Weierstrass classes might fuse under tensor equivalence. We shall give computable invariants to distinguish the tensor equivalence classes and an efficient algorithm for testing equivalence. As an illustration of the weaker equivalence we show that there are only finitely many equivalence classes of $3 \times n \times 2$ tensors for each n (whereas, for Kronecker–Weierstrass equivalence, there are infinitely many classes).

The problem of finding invariants to distinguish the tensor equivalence classes has previously been considered by Ja'Ja' [3] and our approach is similar to his. Unfortunately there is an error in section 3 of [3] and the invariants claimed in that paper are not distinguishing invariants of each equivalence class and so Theorem 2, 3, and 4 are false. However section 2 of [3] contains correct results which we shall build on.

2. TENSOR EQUIVALENCE

Any $m \times n \times 2$ tensor can be specified by two $m \times n$ matrices G_1 and G_2 as indicated in section 1 and these define a homogeneous pencil $\mu G_1 + \lambda G_2$ (μ and λ are indeterminates). The Kronecker–Weierstrass theory shows how this pencil is made up of a regular part and a singular part. Our concern here is with regular pencils ($m = n$ and $\det(\mu G_1 + \lambda G_2) \neq 0$). It was shown in section 2 of [3] that the tensor equivalence theory for the singular part of a pencil is no different to that for the Kronecker–Weierstrass equivalence theory; and, therefore, once the tensor equivalence theory is worked out for regular pencils we can immediately obtain it for general pencils.

Much of the discussion concerns homogeneous polynomials in μ, λ and we preface it with a remark about when two such polynomials are "the same". We regard two polynomials $f(\mu, \lambda), g(\mu, \lambda)$ as essentially the same polynomial if they are projectively equivalent, that is $f(\mu, \lambda) = kg(\mu, \lambda)$ for some non-zero constant k. In particular each linear polynomial $a\mu + b\lambda$ determines and is determined by a unique ratio a/b (the ratio ∞ corresponds to $b = 0$).

As in [3] we let $D_k(\mu, \lambda)$ be the greatest common divisor of all minors of order k in the $n \times n$ regular pencil $\mu G_1 + \lambda G_2$. Then the classical homogeneous invariant polynomials are defined by

$$i_k(\mu, \lambda) = \frac{D_{n-k+1}(\mu, \lambda)}{D_{n-k}(\mu, \lambda)}, \qquad 1 \leq k \leq n.$$

(It can be shown that these quotients are indeed polynomials and, moreover, that $i_k(\mu, \lambda)$ divides $i_{k-1}(\mu, \lambda)$.) The invariant polynomials factor into powers of, say r, distinct irreducible polynomials

$$i_1(\mu,\lambda) = \phi_1(\mu,\lambda)^{\tau_{11}}\phi_2(\mu,\lambda)^{\tau_{12}}\dots\phi_r(\mu,\lambda)^{\tau_{1r}}$$
$$i_2(\mu,\lambda) = \phi_1(\mu,\lambda)^{\tau_{21}}\phi_2(\mu,\lambda)^{\tau_{22}}\dots\phi_r(\mu,\lambda)^{\tau_{2r}}$$
$$\dots\dots\dots$$
$$i_n(\mu,\lambda) = \phi_1(\mu,\lambda)^{\tau_{n1}}\phi_2(\mu,\lambda)^{\tau_{n2}}\dots\phi_r(\mu,\lambda)^{\tau_{nr}}$$

Of course $\tau_{st} \leq \tau_{s-1,t}$ for each relevant s,t and, to avoid trivialities, we take $\tau_{11},\tau_{12},\dots,\tau_{1r} \neq 0$, although neither of these facts plays a significant part in what follows.

The invariant polynomials (or their irreducible factorisations) characterise the Kronecker–Weierstrass equivalence class of the pencil $\mu G_1 + \lambda G_2$. We must investigate these factorisations under the more general tensor equivalence. To this end let (H_1,H_2) be any pair of $n \times n$ matrices which define a tensor equivalent to the one defined by (G_1,G_2). Then there exist non-singular matrices P,Q and a non-singular matrix $\binom{x_{11}\ x_{12}}{x_{21}\ x_{22}}$ such that

$$PH_1Q = x_{11}G_1 + x_{12}G_2 \qquad \text{and} \qquad PH_2Q = x_{21}G_1 + x_{22}G_2$$

The pencil $\mu PH_1Q + \lambda PH_2Q$ has the same invariant polynomials as $\mu H_1 + \lambda H_2$. On the other hand

$$\mu PH_1Q + \lambda PH_2Q = (\mu x_{11} + \lambda x_{21})G_1 + (\mu x_{12} + \lambda x_{22})G_2$$

and therefore the invariant polynomials for $\mu H_1 + \lambda H_2$ are obtained from those of $\mu G_1 + \lambda G_2$ by replacing μ by $\mu x_{11} + \lambda x_{21}$ and λ by $\mu x_{12} + \lambda x_{22}$; and their factorisations are

$$\phi_1(\mu x_{11} + \lambda x_{21}, \mu x_{12} + \lambda x_{22})^{\tau_{k1}}\dots\phi_r(\mu x_{11} + \lambda x_{21}, \mu x_{12} + \lambda x_{22})^{\tau_{kr}} \qquad (1 \leq k \leq n).$$

As a consequence we see that the family of column vectors

$$c_1 = \begin{pmatrix}\tau_{11}\\ \vdots \\ \tau_{n1}\end{pmatrix}, c_2 = \begin{pmatrix}\tau_{12}\\ \vdots \\ \tau_{n2}\end{pmatrix}, \dots, c_r = \begin{pmatrix}\tau_{1r}\\ \vdots \\ \tau_{nr}\end{pmatrix}$$

remains invariant under tensor equivalence.

From now on we shall take the field F to be algebraically closed so that the irreducible polynomials $\phi_i(\mu,\lambda)$ are linear, say

$$\phi_i(\mu,\lambda) = \alpha_i\mu + \beta_i\lambda \qquad \text{for some ratio} \quad \rho_i = \alpha_i/\beta_i.$$

The corresponding polynomials for the equivalent pencil $\mu H_1 + \lambda H_2$ are

$$\phi_i(\mu x_{11} + \lambda x_{21}, \mu x_{12} + \lambda x_{22}) = (\alpha_i x_{11} + \beta_i x_{12})\mu + (\alpha_i x_{21} + \beta_i x_{22})\lambda$$

and are determined by the ratios

$$\frac{\alpha_i x_{11} + \beta_i x_{12}}{\alpha_i x_{21} + \beta_i x_{22}} = \frac{x_{11}\rho_i + x_{12}}{x_{21}\rho_i + x_{22}}$$

Summing up what we have obtained so far, we know that each regular pencil $\mu G_1 + \lambda G_2$ determines a family of columns $c_1, c_2, \ldots, c_r$ and corresponding distinct ratios $\rho_1, \rho_2, \ldots, \rho_r$. Moreover, if $\mu H_1 + \lambda H_2$ is an equivalent pencil, it determines the same family of columns and the associated ratios have the form $(x_{11}\rho_i + x_{12})/(x_{21}\rho_i + x_{22})$ for some nonsingular matrix $\binom{x_{11}\ x_{12}}{x_{21}\ x_{22}}$.

At this point it is convenient to adopt a notation which recognises that the r invariant columns may not be distinct. We describe the r invariant columns by the *column signature* $c_1^{m_1} c_2^{m_2} \ldots c_t^{m_t}$ which indicates that m_1 columns are equal to c_1, m_2 columns are equal to c_2, and so on; thus $\sum m_i = r$. Moreover we order the columns lexicographically so that $c_1 > c_2 > \cdots > c_t$.

Then, of the r ratios $\rho_1, \ldots, \rho_r$, a set R_1 of m_1 of them is associated with column c_1, a set R_2 of m_2 of them is associated with column c_2, and so on. The sequence $(R_1, R_2, \ldots, R_t)$ will be called the *ratio signature* of the pencil. With these notations we may state the main result on tensor equivalence of regular pencils.

THEOREM *Two regular pencils of $n \times n$ matrices are tensor equivalent if and only if*

(a) *they have the same column signature, and*
(b) *for some non-singular matrix $\binom{x_{11}\ x_{12}}{x_{21}\ x_{22}}$ the mapping $\rho \to (x_{11}\rho + x_{12})/(x_{21}\rho + x_{22})$ maps the ratio signature of one pencil onto the ratio signature of the other.*

Proof The discussion above has shown that two tensor equivalent pencils must fulfil (a) and (b). Conversely suppose that (a) and (b) hold for two pencils $\mu G_1 + \lambda G_2$ and $\mu H_1 + \lambda H_2$. The first of these is tensor equivalent to the pencil $\mu(x_{11}G_1 + x_{12}G_2) + \lambda(x_{21}G_1 + x_{22}G_2)$. The latter pencil has, by the remarks above, the same column signature and ratio signature as $\mu H_1 + \lambda H_2$. But these two signatures determine completely the invariant factors of a pencil and so the pencils $\mu(x_{11}G_1 + x_{12}G_2) + \lambda(x_{21}G_1 + x_{22}G_2)$ and $\mu H_1 + \lambda H_2$ are equivalent in the Kronecker–Weierstrass sense. Consequently $\mu G_1 + \lambda G_2$ and $\mu H_1 + \lambda H_2$ are tensor equivalent.

3. EQUIVALENCE CLASS REPRESENTATIVES

In the Kronecker–Weierstrass equivalence theory there is a natural representative of each equivalence class (the Kronecker canonical form, see [2]). For the tensor equivalence theory there does not seem to be a natural representative in most cases. To obtain a representative we would have to choose some ratio signature in an orbit of ratio signatures under the group PGL$(2,F)$ of all transformations $z \to (x_{11}z + x_{12})/(x_{21}z + x_{22})$ with $x_{11}x_{22} \neq x_{12}x_{21}$. This group is a sharply triply transitive group in its action on the set of all ratios $F \cup \{\infty\}$ and consequently given any triple of ratios (α, β, γ) there is a unique transformation in PGL$(2,F)$ which maps this triple to the triple $(\infty, 0, 1)$.

If R_1, R_2, R_3 all had size 1 we could take a representative with $R_1 = \{\infty\}$, $R_2 = \{0\}$, $R_3 = \{1\}$. Since the only element of PGL$(2,F)$ which fixes the triple $(\infty, 0, 1)$ is the identity no further conditions can be placed on the ratio signature. For other cases there is not such an obvious triple of ratios to transform to $(\infty, 0, 1)$ and some artificiality has to be introduced. One possibility would be to totally order the sequences of subsets of F in some way (for example, any total order on F induces an order on subsets of F and hence on sequences of subsets of F by lexicographic rules). As an example of how to use such an order suppose that $|R_1| \geq 3$ and we decided to transform a triple of ratios in R_1 to $(\infty, 0, 1)$. There are $m_1(m_1 - 1) \times (m_2 - 2)$ triples in R_1, any of which could be mapped to $(\infty, 0, 1)$; we would choose the one which minimised the resulting ratio signature. Further details of this approach are given in [1].

When $n = 3$ the triple transitivity of PGL(2,F) allows a complete enumeration of the tensor inequivalent regular pencils. The restrictions $\sum_{i,j} \tau_{ij} = 3$ and $\tau_{st} \leq \tau_{s-1,t}$ show that the only possible column signatures are

$$\text{(i)} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(iv)} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\text{(v)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^3, \quad \text{(vi)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and to each column signature there corresponds just one equivalence class since each ratio signature contains at most 3 distinct ratios and these can be taken as $\infty, 0, 1$. Equivalence class representatives are now easily found. For example, the pencil

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda + \mu \end{pmatrix}$$

represents case (v) (this case arises whenever $\det(\mu G_1 + \lambda G_2)$ is multiplicity-free and so is much the most common).

In conjunction with [3] it follows that there are only a finite number of equivalent $2 \times 3 \times n$ tensors.

For regular pencils with $n > 3$ it is more useful to have an efficient procedure for testing equivalence of ratio signatures under the action of PGL(2,F). A test which operates by computing equivalence class representatives would require $O(n^4 \log n)$ arithmetic operations if it used the suggestions for equivalence class representatives above. In the final section we give a test for tensor equivalence whose cost is only $O(n^2 \log n)$ arithmetic operations assuming that the Kronecker canonical forms have been computed.

4. TESTING EQUIVALENCE

In this section we give an algorithm to test condition (b) of the theorem. Let $\mathbf{R} = (R_1, R_2, \ldots, R_t)$ and $\mathbf{S} = (S_1, S_2, \ldots, S_t)$ be two ratio signatures with $|R_i| = |S_i|$,

$i = 1,2,\ldots,t$ and with $\sum|R_i| = r \le n$. We shall show how to find, in $O(n^2\log n)$ operations, an element $\sigma \in \mathrm{PGL}(2,F)$ which maps **R** to **S** (or prove that no such element exists).

Let r_1 be one of the points of R_1 and let $S_1 = \{s_1,\ldots,s_a\}$. Consider the following elements of PGL$(2,F)$:

$$\alpha : z \to \frac{1}{z-r_1}, \qquad \beta_k : z \to \frac{1}{z-s_k}, \qquad k = 1,2,\ldots,a$$

Note that $r_1\alpha = s_k\beta_k = \infty$. If there is an element $\sigma \in \mathrm{PGL}(2,F)$ which maps **R** to **S** then it must map r_1 to some s_k. But then $\infty\alpha^{-1}\sigma\beta_k = r_1\sigma\beta_k = s_k\beta_k = \infty$ so that $\alpha^{-1}\sigma\beta_k$ would be affine (of the form $z \to uz+v$) and it would map each $R_i\alpha$ to $S_i\beta_k$. Conversely, any affine transformation which maps each $R_i\alpha$ to $S_i\beta_k$ has the form $\alpha^{-1}\sigma\beta_k$ where $\sigma \in \mathrm{PGL}(2,F)$ maps **R** to **S**. Thus, to construct σ (if it exists) we have to consider, for each $k = 1,2,\ldots,a$, whether there exists an affine transformation mapping $\mathbf{R}\alpha$ to $\mathbf{S}\beta_k$. If, for a fixed k, we can do this in $O(n\log n)$ operations then we shall have a test for condition (b) in $O(n^2\log n)$ operations.

So it remains to consider the following problem. If $\mathbf{R} = (R_1,R_2,\ldots,R_t)$ and $\mathbf{S} = (S_1,S_2,\ldots,S_t)$ are two ratio signatures, does there exist an affine transformation which maps **R** to **S**. We may presume that the point ∞ occurs in neither signature for, if it did occur, it would have to occur in corresponding sets R_i, S_i in order for σ to exist and could be deleted from both. The affine transformations form a subgroup of PGL$(2,F)$ and we use this fact to reduce the search for σ to affine transformations of a more specific form.

Let $R = \bigcup R_i$, $S = \bigcup S_i$, put $c = (1/r)\sum_{x\in R} x$, $d = (1/r)\sum_{x\in S} x$, and define $\gamma : z \to z-c$, $\delta : z \to z-d$. It is easy to check that, if σ maps **R** to **S**, then $c\sigma = d$ and $\gamma^{-1}\sigma\delta$ maps $\mathbf{R}\gamma$ to $\mathbf{S}\delta$; then, because $c\gamma = d\delta = 0$, $\gamma^{-1}\sigma\delta$ has the form $z \to pz$. Hence, by replacing **R**, **S** by $\mathbf{R}\gamma$, $\mathbf{S}\delta$ we may restrict our search to σ of the form $z \to pz$.

For a further reduction we compute

$$f = \max\{|z| \mid z \in R\}, \qquad g = \max\{|z| \mid z \in S\},$$

let ϵ be the transformation $z \to (f/g)z$, and replace **S** by $\mathbf{S}\epsilon$. Any affine transformation which maps **R** to (the new) **S** must map points of maximum modulus of **R** into points of maximum modulus in **S**. But these two maxima are now equal and so the only transformations that can map **R** to **S** have the form $z \to ze^{i\theta}$; they correspond to rotations about the origin in the Argand diagram.

We shall represent every point in $R = \bigcup R_i$ by a triple (θ,μ,i) consisting of its argument θ, modulus μ, and *nametag* (the index i of the set R_i which contains it). Then we sort these triples so that they are listed $(\theta_1,\mu_1,i_1),(\theta_2,\mu_2,i_2),\ldots,(\theta_r,\mu_r,i_r)$ in non-decreasing order of argument, with triples with equal arguments coming in non-decreasing order of modulus. Similarly we represent and arrange the points of $S = \bigcup S_i$ as a list $(\phi_1,\nu_1,j_1),\ldots,(\phi_r,\nu_r,j_r)$. Any rotation which maps **R** to **S** must, for some h, map each $\mu_k e^{i\theta_k}$ onto $\nu_{k+h}e^{i\theta_{k+h}}$ and have $i_k = j_{k+h}$ (here, and subsequently, all subscripts are reduced modulo r to lie between 1 and r). Thus a necessary and sufficient condition for such a rotation to exist is

$$\theta_{k+1} - \theta_k = \phi_{k+h+1} - \phi_{k+h}$$
$$\mu_k = \nu_{k+h}$$
$$i_k = j_{k+h}$$

for $k = 1, 2, \ldots, r$ (and the first equation is interpreted modulo 2π).

Now put $\alpha_R = (\theta_{k+1} - \theta_k, \mu_k, i_k)$ and $\beta_k = (\phi_{k+1} - \phi_k, \nu_k, j_k)$. Then, if we consider the two sequences $(\alpha_1, \ldots, \alpha_r)$ and $(\beta_1, \ldots, \beta_r)$, the necessary and sufficient condition above is that some cyclic shift of one sequence produces the other. But this condition can be formulated as a pattern matching problem: it is that $(\alpha_1, \ldots, \alpha_r)$ occurs as a substring of $(\beta_1, \ldots, \beta_r, \beta_1, \ldots, \beta_{r-1})$. This question can be decided in $O(r)$ operations by the Knuth–Morris–Pratt pattern matching algorithm [4].

The cost of deciding whether an affine transformation mapping **R** to **S** exists is dominated by the sorting step and is $O(r \log r) = O(n \log n)$ as required. This completes the discussion of the algorithm to check condition (b) of the theorem.

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