SPACES OF MATRICES OF BOUNDED RANK

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In this paper we shall consider matrices over a field F and shall prove the following result:

THEOREM. *Let $\mathcal{M}$ be a 2-dimensional space of $m \times n$ matrices with the property that* $\operatorname{rank}(X) \leq k < |F|$ *for every $X \in \mathcal{M}$. Then there exist two integers r, s, $0 \leq r, s \leq k$ with $r + s = k$, and two non-singular matrices P, Q such that, for all $X \in \mathcal{M}$, PXQ has the form*

$$\left[\begin{array}{c|c} r \times s & \\ \hline & 0 \end{array}\right]$$

Notice that a matrix of the above form necessarily has rank at most k and so, apart from the restriction $|F| > k$, our theorem essentially characterises such 2-dimensional subspaces. We may interpret the matrices as the matrices of linear transformations or of bilinear forms and this gives the following two equivalent forms of the theorem valid for finite dimensional vector spaces:

COROLLARY 1. *Let $\mathcal{M}$ be a 2-dimensional space of linear transformations from a vector space U to a space V such that* $\operatorname{rank}(X) \leq k < |F|$ *for every $X \in \mathcal{M}$. Then there exist subspaces $U_0 \leq U$, $V_0 \leq V$ such that* $[U\colon U_0] + \dim(V_0) = k$ *and $U_0 X \leq V_0$ for every $X \in \mathcal{M}$.*

COROLLARY 2. *Let $\mathcal{M}$ be a 2-dimensional space of bilinear forms on $U \times V$ to F such that* $\operatorname{rank}(X) \leq k < |F|$ *for every $X \in \mathcal{M}$. Then there exist subspaces $U_0 \leq U$, $V_0 \leq V$ such that $[U\colon U_0] + [V\colon V_0] = k$ and $(U_0, V_0)X = \{0\}$ for all $X \in \mathcal{M}$.*

Our interest in this result stems from its application to computational complexity [**1**] but conditions like those of the theorem have been studied before [**2, 3**]. Indeed in [**3**] the conclusion of our theorem is proved under the stronger assumptions $\operatorname{rank}(X) = k$ for all $X \in \mathcal{M}$ and F is algebraically closed. However our result, besides being more general, has a shorter proof.

We note that, in general, the conditions in the theorem cannot be weakened. The condition $|F| > k$ is necessary on account of the space generated by $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ over $GF(2)$ and the condition $\dim(\mathcal{M}) = 2$ is necessary on account of the 3-dimensional space of 3×3

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skew-symmetric matrices over any field. In each of these examples all the matrices clearly have rank at most 2 but a small amount of calculation reveals that the spaces do not satisfy the conclusion of the theorem.

Proof of Theorem. We work with a basis of $\mathcal{M}$ (consisting of two matrices A, B) and show that row and column operations may be applied to bring A and B simultaneously to the required form. We shall assume, by induction on k, that the result is already established for smaller values (the base, $k=0$, of the induction is vacuous; the case $k=1$ can also be proved easily and in this case $\mathcal{M}$ can be of arbitrary dimension).

We may assume that $\mathcal{M}$ contains a matrix of rank precisely k and by row and column operations we may take it to be

$$A=\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$$

with A_0 a $k\times k$ non-singular matrix. Let B complete $\{A\}$ to a basis of $\mathcal{M}$ and let

$$B=\begin{bmatrix} C & D \\ E & G \end{bmatrix}$$

where C is a $k\times k$ matrix.

Then $G=0$. For suppose it contained a non-zero entry x. Then by performing certain row and column operations which leave A invariant we may reduce all the other entries in the row and column containing x to zero. It is then clear that $\operatorname{rank}(\lambda A-B)\geq \operatorname{rank}(\lambda A_0-C)+1$ for all λ. But, as A_0 is non-singular, the equation $\det(\lambda A_0-C)=0$ has at most k distinct solutions. So, as $|F|>k$, we can choose λ so that λA_0-C is non-singular and then $\operatorname{rank}(\lambda A-B)\geq k+1$, a contradiction.

Now let $u=\operatorname{rank}(D)$, $v=\operatorname{rank}(E)$. Apply row and column operations to bring B to the form

$$\left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{C}} & D_0 & 0 \\ \cline{3-4} \multicolumn{2}{c|}{} & 0 & 0 \\ \hline E_0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \cline{1-2} 0 & 0 & \multicolumn{2}{c}{} \end{array}\right]$$

where D_0, E_0 are non-singular $u\times u$, $v\times v$ matrices respectively. These row and column operations will affect A but it will still have the form $\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ for some possibly different $k\times k$ non-singular matrix A_0.

We partition the matrix B as

$$B=\left[\begin{array}{c|c|c|c} & & D_0 & 0 \\ \cline{2-4} & Y & 0 & 0 \\ \hline E_0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \cline{1-2} 0 & 0 & \multicolumn{2}{c}{} \end{array}\right]$$

and we let X denote the submatrix of A which corresponds to Y. Since D_0, E_0 have ranks u, v respectively it is clear that, for any non-zero λ and for any μ, $\text{rank}(\mu A+\lambda B)\geqslant u+v+\text{rank}(\mu X+\lambda Y)$. Hence $\text{rank}(\mu X+\lambda Y)\leqslant k-u-v$ for all non-zero λ. If $u=0$ then the result is already established with $r=0$, $s=k$. Otherwise we have $\text{rank}(\mu X+\lambda Y)\leqslant k-u-v<k$ for all μ and all non-zero λ. If this inequality could be established also in the case $\lambda=0$ then the induction hypothesis would apply to $\langle X, Y\rangle$. Then there would be row and column operations which simultaneously reduced X, Y to the form

$$\left[\begin{array}{c|c} p\times q & \\ \hline & 0 \end{array}\right]$$

with $p+q=k-u-v$. These row and column operations could be induced by row and column operations on A and B and the theorem would follow with $r=p+u$, $s=q+v$.

Thus, to complete the proof, we have to show that $\text{rank}(X)\leqslant h=k-u-v$. Suppose that this is not so. Then let Z be an $(h+1)\times(h+1)$ non-singular minor of X and let W be the corresponding minor of Y. Then, because $\mu X+Y$ has rank at most h for all μ, we have

$$\det(\mu Z+W)=0$$

for all $\mu\in F$. But since Z is non-singular this equation has at most $h+1\leqslant k$ distinct roots and this contradicts $|F|>k$.

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