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A Problem of Westwick on k-Spaces

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Sets of $n \times n$ matrices whose linear span contains only matrices of rank $n-1$ and 0 are investigated. To within a natural equivalence they are characterised for $n \leqslant 6$. Partial results are obtained for general n.

A *k-space* as defined by Westwick [7] is a vector space whose elements are linear transformations from a space V into a space W all of which, except for the zero transformation, have the same rank k. k-spaces have since been investigated by Beasley [3, 4] who used them to obtain results on homomorphisms which preserve rank, and by Atkinson and Westwick [2] who produced infinitely many examples for certain fixed k, V and W. k-spaces have also found applications in algebraic computational complexity [6].

We shall adopt the language of matrices rather than linear transformations. Consequently two k-spaces $\mathscr{X}, \mathscr{Y}$ of $m \times n$ matrices are regarded as *equivalent* if there exist non-singular matrices P, Q (which correspond to changes of basis in V and W) such that

$$\mathscr{Y} = \{PXQ \mid X \in \mathscr{X}\}$$

Throughout the paper we shall take the ground field to be algebraically closed. It is rather obvious that an n-space of $n \times n$ matrices cannot have dimension greater than 1. In view of this Westwick drew

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special attention to the problem of characterising $(n-1)$-spaces of $n \times n$ matrices. We shall call these spaces *exactly singular* spaces; their elements are of rank precisely $n-1$ or zero. Using the dimension theorem of algebraic geometry Westwick showed that exactly singular spaces must have dimension at most 4. He also proved that for $n = 2, 3, 4$ their dimensions are at most $2, 3, 2$ respectively; the last of these results required quite a tricky argument. Finally, for each odd n, he gave an example of a 3-dimensional exactly singular space.

In this paper we shall introduce a new technique for studying exactly singular spaces. We use it to prove a general result on partitioned matrices and to settle the cases $n = 5, 6$. Before this however we shall summarise some of the more elementary facts about exactly singular spaces.

It is evident that a 1-dimensional exactly singular space is equivalent to the space generated by the $n \times n$ matrix $I_{n-1} \oplus 0_1$ and so it is essentially unique. A 2-dimensional exactly singular space determines, via a fixed basis A, B, a homogeneous matrix pencil $\lambda A + \mu B$. The Kronecker theory of pencils shows that there is an equivalent pencil of the form $L_p^T \oplus L_q$ with $p + q = n - 1$ (see [5] for the notation and theory). Note that this pencil has $(\mu^p, \mu^{p-1}\lambda, \ldots, \lambda^p, 0, 0, \ldots, 0)$ as a left null vector and $(0, 0, \ldots, 0, \mu^q, \mu^{q-1}\lambda, \ldots, \lambda^q)^T$ as a right null vector. For each n there are exactly n inequivalent 2-dimensional exactly singular spaces, one for each p in $0 \leqslant p \leqslant n-1$.

Westwick defined a k-space $\mathscr{X}$ to be *essentially decomposable* if for some i, j with $i + j = k$ every matrix of an equivalent space $\mathscr{Y}$ had the form

$$\left[\begin{array}{c|c} i \times j & \\ \hline & 0 \end{array}\right]$$

He showed (Corollary 2.2 of [7])

Lemma 1 *Every decomposable exactly singular space has dimension at most* 2.

We now define some notation which will be used throughout this paper.

Suppose $\mathscr{X}$ is an exactly singular space of $n \times n$ matrices. Let $A_1, A_2, \ldots$ be a basis of $\mathscr{X}$, let $x_1, x_2, \ldots$ be independent transcendentals over the ground field and define

$$X = \sum x_i A_i$$

a generic point of the linear variety $\mathscr{X}$. Since rank $X = n - 1$ X has both a left null row-vector $\mathbf{u}$ and a right null column-vector $\mathbf{v}$. The components of these vectors may be taken to be homogeneous polynomials in $x_1, x_2, \ldots$. We shall choose $\mathbf{u}$ and $\mathbf{v}$ so that the total degrees p and q of their components are minimal. It is easy to see that any other left null row-vector with polynomial components must be a polynomial multiple of $\mathbf{u}$ (similarly for right null column-vectors).

Since $\mathbf{u}P^{-1}PX = 0$ for any non-singular matrix P, $\mathbf{u}P^{-1}$is the minimal left null vector for PX. Thus the null vectors of spaces equivalent to $\mathscr{X}$ are related to $\mathbf{u}$ and $\mathbf{v}$ by applying a non-singular matrix with entries in the ground field. Moreover the generic matrix X depends on the basis chosen for $\mathscr{X}$. Any other basis will give a generic matrix whose transcendentals $x'_1, x'_2, \ldots$ are related to x_1, $x_2, \ldots$ by a non-singular linear substitution $\mathbf{x}' = \mathbf{x}T$. Equivalences and changes of basis may often be used to simplify the components of $\mathbf{u}$ and $\mathbf{v}$.

LEMMA 2 *If* $\dim \mathscr{X} \geqslant 2$ *then*

(i) $p + q = n - 1$,

(ii) *every specialisation* $\mathbf{x} \to \xi$, *with* $\xi \neq 0$, *specialises* $\mathbf{u}$ *and* $\mathbf{v}$ *to non-zero vectors,*

(iii) *all* 2-*dimensional subspaces of* $\mathscr{X}$ *are equivalent,*

(iv) *every specialisation* $\mathbf{x} \to \xi$, *where* ξ *is a generic point of a* 2-*dimensional linear variety, specialises* $\mathbf{u}$ (*or* $\mathbf{v}$) *to a vector whose polynomial components span a linear space of dimension* $p + 1$ (*or* $q + 1$) *over the ground field.*

Proof The adjoint of X has polynomial components of degree $n - 1$ (or 0). The adjoint has rank 1, since rank $X = n - 1$, and so may be written

$$\operatorname{adj} X = \mathbf{r} \cdot \mathbf{s}$$

where $\mathbf{r}$ is a column vector and $\mathbf{s}$ is a row vector. Clearly $\mathbf{r}$ and $\mathbf{s}$ may be taken to have polynomial components. From the equation $X \cdot \operatorname{adj} X = 0$ it follows that $\mathbf{r}$ is a right null column-vector for X and so is a polynomial multiple of $\mathbf{v}$. Similarly $\mathbf{s}$ is a polynomial multiple of $\mathbf{u}$. Consequently

$$p + q \leqslant n - 1.$$

Each 2-dimensional subspace of $\mathscr{X}$ has a generic point $\sum \xi_i A_i$ where $\xi_1, \xi_2, \ldots$ are linear expressions in two transcendentals ζ, η. Conversely if $\xi_1, \xi_2, \ldots$ are linear expressions in ζ, η which are not all multiples of a single expression, then $\sum \xi_i A_i$ is a generic point of a 2-dimensional subspace of $\mathscr{X}$.

Now consider a 2-dimensional subspace $\mathscr{Y}$ of $\mathscr{X}$ for which $\mathbf{u}(\xi_1, \xi_2, \ldots)$ (and $\mathbf{v}(\xi_1, \xi_2, \ldots)$) are non-zero (so their components are of degree p (and q) in ζ and η). $\mathscr{Y}$ is also an exactly singular space and, according to the Kronecker theory, its generic point is equivalent to a matrix

$$L_a^T \oplus L_b$$

where a, b are the degrees of its minimal degree left and right null vectors. Since $\mathbf{u}(\xi_1 \xi_2, \ldots)$ and $\mathbf{v}(\xi_1, \xi_2, \ldots)$ are left and right null vectors of this matrix we have $a \leqslant p, b \leqslant q$, and so

$$n - 1 = a + b \leqslant p + q \leqslant n - 1.$$

This proves statement (i) and also proves that $p = a$, $q = b$ and, neglecting scalar factors, $\operatorname{adj} X = \mathbf{v} \cdot \mathbf{u}$. Statement (ii) follows from this equation; for if a specialisation $\mathbf{x} \to \xi$ specialises $\mathbf{u}$ to 0 it specialises $\operatorname{adj} \mathrm{X}$ to 0 and therefore $\operatorname{rank}(\sum \xi_i A_i) < n - 1$; hence $\sum \xi_i A_i = 0$ and hence $\xi = 0$. Statement (iii) also follows for the above argument shows that the Kronecker form of every 2-dimensional subspace $\mathscr{Y}$ is defined by $L_p^T \oplus L_q$.

Finally to prove statement (iv) note that the specialisation in question specialises $\mathbf{u}$ to a vector $\mathbf{u}^*$ whose components are polynomials in two variables ζ and η of total degree p. When the 2-dimensional linear variety is transformed to Kronecker canonical form the vector $\mathbf{u}^*$ is transformed by an $n \times n$ non-singular matrix in the ground field to a vector

$$(\zeta^p, \zeta^{p-1}\eta, \ldots, \zeta\eta^{p-1}, \eta^p, 0, 0, \ldots,)$$

Since the components of this vector span a linear space of dimension $p + 1$ the same is true of the components of $\mathbf{u}^*$.

PROPOSITION *Let $\mathscr{X}$ be an exactly singular space for which*

$$X = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

is a non-trivial partitioning with A and B both square. Then

(i) *at least one of A and B is singular,*

(ii) *if A and B are both singular then* $\dim\mathscr{X} \leqslant 1$,
(iii) *in any case*, $\dim\mathscr{X} \leqslant 2$.

Proof (i) $0 = \det X = \det A \cdot \det B$ and so one of $\det A$ and $\det B$ is zero.

(ii) Suppose that both A and B are singular and that $\dim\mathscr{X} \geqslant 2$. Clearly both are generic matrices of exactly singular spaces of, say, $a \times a$ and $b \times b$ matrices and these spaces have dimension $\dim\mathscr{X}$. Let $\mathbf{s}$ be a minimal left null vector of A. Then $(\mathbf{s}, 0, \ldots, 0)$ is a left null vector of X and by Lemma 2 its components are of degree at most $a - 1$. Hence $p \leqslant a - 1$. Similarly $q \leqslant b - 1$. Then $n - 1 = p + q \leqslant a - 1 + b - 1 = n - 2$, a contradiction.

(iii) Suppose, without loss in generality, that A is singular; again A is a generic matrix for an exactly singular space. Let $\mathbf{t}$ be a right null vector of A and consider the product

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}\begin{bmatrix} \mathbf{t} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c\mathbf{t} & B \end{bmatrix}$$

(here the second and third matrices are of size $n \times (b + 1)$). By Sylvester's inequality

$$\operatorname{rank}\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} + \operatorname{rank}\begin{bmatrix} \mathbf{t} & 0 \\ 0 & I \end{bmatrix} \leqslant n + \operatorname{rank}\begin{bmatrix} C\mathbf{t} & B \end{bmatrix}.$$

Consider an arbitrary specialisation $\mathbf{x} \to \xi \neq 0$. In such a specialisation $\mathbf{t}$ remains non-zero (Lemma 2) and so

$$\operatorname{rank}\begin{bmatrix} C\mathbf{t} & B \end{bmatrix} \geqslant n - 1 + b + 1 - n = b.$$

Hence $\operatorname{rank}[C\mathbf{t}\ B] = b$ under this specialisation.

The variety $\mathscr{Y}$ whose generic point is $[C\mathbf{t}\ B]$ (in affine $b(b + 1)$-space) is irreducible and of dimension at most $\dim\mathscr{X}$. However since B is a generic point of a variety of the same dimension as $\mathscr{X}$ we have $\dim\mathscr{Y} = \dim\mathscr{X}$. But $\mathscr{Y}$ has no non-zero intersection with the variety of $b \times (b + 1)$ matrices of rank less than b and this latter variety has dimension $b(b + 1) - 2$ ([7] Theorem 2.1). Hence $\dim\mathscr{X} = \dim\mathscr{Y} \leqslant 2$.

LEMMA 3 *If one of p and q is zero then* dim $\leqslant 2$.

Proof If $p = 0$ then every matrix of $\mathscr{X}$ has a common left null vector. Therefore $\mathscr{X}$ is decomposable (with $i = n - 1$, $j = 0$) and Lemma 1 applies.

LEMMA 4 *If one of p and q is 1 then* $\dim \mathscr{X} \leqslant 3$ *with equality only if* $n = 3$ *when, to within equivalence,* $\mathscr{X}$ *is unique.*

Proof It is known that, when $n = 3$, there is precisely one exactly singular space of dimension 3 or more ([1], p. 313). Therefore to prove the lemma it suffices to consider a 3-dimensional exactly singular space $\mathscr{X}$, assume that $p = 1$, and deduce that $n = 3$.

The components of the left null vector $\mathbf{u}$ of X are linear expressions in variables x_1, x_2, x_3, and so the components span a space of dimension at most 3. Replacing $\mathscr{X}$ by an equivalent space we may take $\mathbf{u} = (u_1, u_2, u_3, 0, 0, \ldots, 0)$. From $uX = 0$ we find that

$$x = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

where Y is a $3 \times n$ matrix satisfying $\mathbf{u}Y = 0$. The linear space $\mathscr{Y}$ of $3 \times n$ matrices determined by Y is a 2-space. It cannot be decomposable or $\mathscr{X}$ itself would be decomposable contradicting Lemma 1. By p. 314 of [1] $\mathscr{Y}$ is equivalent to a space whose matrices have entries in the first 3 columns only. Thus $\mathscr{X}$ is equivalent to a space of partitioned matrices of the form

$$\left[\begin{array}{c|c} 3 \times 3 & 0 \\ \hline & \end{array}\right]$$

By the proposition this partition is degenerate and so $n = 3$.

This result allows Westwick's theorem on $n = 4$ to be deduced immediately; since $p + q = 3$ one of p and q is 0 or 1. We turn now to other small values of n.

Let $n = 5$ or 6. Then $p + q = 4$ or 5 and so one of p and q, p say, is at most 2. However the cases $p = 0, 1$ are dealt with by the results above and if the exactly singular space has dimension 3 or more they do not arise. So in our discussion of $n = 5, 6$ we may take $p = 2$.

Let $(u_1, u_2, \ldots)$ be the left null vector of X. Each u_i is a quadratic form in the variables $x_1, x_2, \ldots$. We shall consider the space generated by these quadratic forms.

LEMMA 5 *If* $\dim \mathscr{X} \geqslant 3$, *then* $\dim\langle u_1, u_2, \ldots \rangle > 4$.

Proof We replace $\mathscr{X}$ by an equivalent space chosen so that the left null vector of the new $\mathscr{X}$ is $(u_1, u_2, \ldots, u_d, 0, \ldots, 0)$ with all $u_1, u_2, \ldots, u_d$ independent. Then we have

$$X = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

where Y is a $d \times n$ matrix satisfying

$$(u_1, u_2, \ldots, u_d)Y = 0.$$

Thus Y determines a space $\mathscr{Y}$ of matrices all of which have rank $d - 1$ (or 0). Notice that $\mathscr{Y}$ cannot be decomposable or else $\mathscr{X}$ would also be decomposable and, as $\dim \mathscr{X} \geqslant 3$, this is impossible. Now suppose for a contradiction that $d \leqslant 4$. Thus $\mathscr{Y}$ is an indecomposable space of $d \times n$ matrices all of the same rank $d - 1 \leqslant 3$. Such spaces were determined as consequences of Theorem C of [1] and all of them had linear left null vectors; this is the required contradiction.

THEOREM A *All exactly singular spaces of* 5×5 *matrices have dimension at most* 3. *To within equivalence there is precisely one* 3-*dimensional exactly singular space of* 5×5 *matrices.*

Proof We continue to use the notation introduced above. Suppose first that $\mathscr{X}$ is a 4-dimensional exactly singular space of 5×5 matrices. The quadratic forms u_i may be represented by symmetric 4×4 matrices U_i since

$$u_i = (x_1, x_2, x_3, x_4)U_i \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

By the previous lemma $\langle U_1, \ldots, U_5 \rangle$ is 5-dimensional and, by [7] Theorem 2.1, must contain a non-zero matrix of rank 1 or 2. By a non-singular change of variables we may take the corresponding form to be either x_4^2 or $x_3 x_4$. If we now form the 3-dimensional subspace of $\mathscr{X}$ obtained by specialising x_4 to 0 we obtain an exactly singular space with a left null vector with at most 4 non-zero components and this contradicts the previous lemma.

To prove the second part of the theorem we suppose that $\mathscr{X}$ is a 3-dimensional exactly singular space of 5×5 matrices. The matrices U_i introduced above are now symmetric 3×3 matrices and generate a 5-dimensional space $\mathscr{U}$. We shall characterise $\mathscr{X}$ by first characterising $\mathscr{U}$.

The entries of every matrix $[u_{ij}] = U \in \mathscr{U}$ satisfy 3 linear relations $u_{ij} = u_{ji}$. However $\mathscr{U}$ is of codimension 4 in the space of all 3×3 matrices and so its matrix entries satisfy one more linear relation which may be written

$$\operatorname{trace}(BU) = 0.$$

But then

$$\operatorname{trace}(B^TU) = \operatorname{trace}(U^TB) = \operatorname{trace}(UB) = \operatorname{trace}(BU) = 0$$

and so trace($(B + B^T)U$) = 0, i.e. we may take B to be symmetric.

Choose a non-singular matrix P such that PBP^T is diagonal with 1's and 0's on the diagonal, replace $U_1, \ldots, U_5$ by $P^{T^{-1}}U_iP$, $i = 1, \ldots, 5$ and change variables so that the matrices $P^{T^{-1}}U_iP$ again represent the components of the null vector. Thus we may take the null vector components to be represented by symmetric matrices U which have trace(BU) = 0 with B diagonal.

We can now deduce that $B = I$. For if rank $B = 1$ or 2 then the matrices of $\mathscr{U}$ have the forms

$$\begin{bmatrix} 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} y & \cdot & \cdot \\ \cdot & -y & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

respectively. In either case if we consider the subspace of $\mathscr{X}$ obtained by specialising x_3 to 0 we find that its null vector has components which span a space of dimension at most 2, contradicting lemma 2.

As a basis for $\mathscr{U}$ we can take

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and then the null vector is

$$\mathbf{u} = \left(x_1x_2, x_2x_3, x_1x_3, x_1^2 - x_2^2, x_2^2 - x_3^2\right).$$

The columns of X are linear functions of x_1, x_2, x_3 and are orthogonal to $\mathbf{u}$. A typical column

$$\left(\sum a_ix_i, \sum b_ix_i, \sum c_ix_i, \sum d_ix_i, \sum e_ix_i\right)^T$$

of X therefore satisfies

$$\begin{aligned} x_1x_2\sum a_ix_i + x_2x_3\sum b_ix_i + x_1x_3\sum c_ix_i \\ + \left(x_1^2 - x_2^2\right)\sum d_ix_i + \left(x_2^2 - x_3^2\right)\sum e_ix_i = 0. \end{aligned}$$

This identity gives rise to 10 linear equations (one for each cubic monomial) in the 15 unknowns $a_1, \ldots, e_3$. The equations are independent and so the solution space is 5-dimensional.

On the other hand the columns of X are linearly independent (over the ground field) since $\mathscr{X}$ is indecomposable. Thus the columns of X

are a basis for the solution space. Different bases correspond to equivalent matrices XP (P nonsingular with entries in the ground field) and so correspond to equivalent spaces. Thus $\mathscr{X}$ is unique.

THEOREM B *Every exactly singular space of* 6×6 *matrices has dimension at most* 2.

Proof We shall maintain the previous notation and shall suppose for a contradiction that $\mathscr{X}$ is a 3-dimensional exactly singular space of 6×6 matrices.

Note first that the space generated by the components of the left null vector $\mathbf{u}$ cannot be 5-dimensional. If it were, then the arguments used in the proof of the previous theorem would show that some space equivalent to $\mathscr{X}$ has generic matrix of the form

$$\left[\begin{array}{c|c} & 0 \\ & 0 \\ 5 \times 5 & 0 \\ & 0 \\ & 0 \\ \hline \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \end{array}\right]$$

and, by the proposition, this is impossible. The components of $\mathbf{u}$ therefore generate the space of all quadratic forms in the variables x_1, x_2, x_3 and $\mathbf{u}$ may be taken to be

$$(x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1).$$

Every column of the generic matrix X is orthogonal to $\mathbf{u}$. The condition that a typical column

$$\left(\sum a_ix_i, \sum b_ix_i, \sum c_ix_i, \sum d_ix_i, \sum e_ix_i, \sum f_ix_i\right)^T$$

be orthogonal to $\mathbf{u}$ gives rise, as in the previous proof, to 10 independent linear equations this time in 18 unknowns. The solution space of this system is 8-dimensional and the following 8 columns are a basis for it:

$$\begin{array}{cccccccc}
0 & 0 & x_3 & x_2 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & x_1 & 0 & 0 \\
-x_2 & 0 & 0 & -x_1 & 0 & 0 & x_3 & 0 \\
0 & -x_3 & 0 & 0 & -x_2 & 0 & 0 & x_1 \\
0 & 0 & -x_1 & 0 & 0 & -x_3 & -x_2 & -x_2
\end{array}$$

The 6 columns of X (which are independent since $\mathscr{X}$ is indecomposable) span a subspace of the 8-dimensional space generated by these 8 columns. Thus we have

$$SP = X$$

where S is the 6×8 matrix displayed above and P is an 8×6 matrix with entries in the ground field. By virtue of this equation we may, for each specialisation of x_1, x_2, x_3, regard P as a linear mapping from the row space of the specialised S to the row space of the specialised X (a member of $\mathscr{X}$).

Considered as a mapping on row vectors P has a 2-dimensional kernel generated say by vectors $\mathbf{e}, \mathbf{f}$ of length 8. We shall show that, for some non-zero specialisation, $\langle \mathbf{e}, \mathbf{f} \rangle$ non-trivially intersects the row space of the specialised S. This will give us a contradiction for the specialised S will have rank at most 5 (the specialised $\mathbf{u}$ is a left null vector) and P will not transform its row space faithfully; thus the corresponding specialised X will have rank at most 4.

A vector $\mathbf{g}$ belongs to the row space of S if and only if it is orthogonal to all the right null vectors of S. These null vectors are spanned by the column vectors

$$\begin{matrix} 0 & x_3 & 0 \\ 0 & 0 & x_1 \\ x_2 & 0 & 0 \\ -x_3 & 0 & 0 \\ 0 & -x_1 & 0 \\ 0 & 0 & -x_2 \\ -x_1 & x_2 & 0 \\ 0 & -x_2 & x_3 \end{matrix}$$

(and these vectors remain independent in all non-zero specialisations). Hence the condition that $\mathbf{g}$ belongs to the row space of S when specialised according to the specialisation $\mathbf{x} \to \xi$ is

$$\begin{aligned} -\xi_1 g_7 + \xi_2 g_3 - \xi_3 g_4 &= 0 \\ -\xi_1 g_5 + \xi_2(g_7 - g_8) + \xi_3 g_1 &= 0 \\ \xi_1 g_2 - \xi_2 g_6 + \xi_3 g_8 &= 0 \end{aligned}$$

Hence the sought for non-zero specialisation can be found provided that the matrix of coefficients G is singular for some non-zero

$\mathbf{g} \in \langle \mathbf{e}, \mathbf{f} \rangle$. Such a matrix G can be found for if $\mathbf{g} = \alpha\mathbf{e} + \beta\mathbf{f}$ then

$$G = \begin{bmatrix} -\alpha e_7 - \beta f_7 & \alpha e_3 + \beta f_3 & -\alpha e_4 - \beta f_4 \\ -\alpha e_5 - \beta f_5 & \alpha e_7 + \beta f_7 - \alpha e_8 - \beta f_8 & \alpha e_1 + \beta f_1 \\ \alpha e_2 + \beta f_2 & -\alpha e_6 - \beta f_6 & \alpha e_8 + \beta f_8 \end{bmatrix}$$
$$= \alpha E + \beta F$$

and because E, F are independent they have a non-zero singular linear combination. This completes the proof.

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