SIAM J. COMPUT.
Vol. 12, No. 4, November 1983

0097-5397/83/1204-0001 $01.25/0

THE RANKS OF $m \times n \times (mn-2)$ TENSORS*

M. D. ATKINSON† AND S. LLOYD†

Abstract. It is shown that there is essentially only one $m \times n \times (mn-2)$ tensor of rank $mn-1$. It is also proved that, except for this tensor, all $m \times n \times p$ tensors with $p \leqq mn-2$ have rank at most $mn-2$. The main tool is Kronecker's theory of matrix pencils which has already been applied directly by Ja'Ja' [SIAM J. Comput., 8 (1979), pp. 443–462] to study the ranks of $m \times n \times 2$ tensors. We show that each nondegenerate $m \times n \times (mn-2)$ tensor is determined by a related $m \times n \times 2$ tensor and apply the Kronecker theory to this related tensor.

Key words. computational complexity, bilinear forms, tensor rank

A classical problem in algebraic computational complexity is to determine the minimal number of nonscalar multiplications required to evaluate some set $\sum_{i,j} \alpha_{ijk} x_i y_j$, $k=1,2,\cdots,p$, of bilinear forms in noncommuting variables $x_1,\cdots,x_m$, $y_1,\cdots,y_n$. This number can be variously described as the complexity of the set, the rank of the defining 3-tensor (α_{ijk}) or as the minimal number of rank 1 matrices whose linear span contains the $m \times n$ matrices $A_k = (\alpha_{ijk})$, $k=1,2,\cdots,p$ [2].

The problem is rather trivial if $p=1$ when the complexity is simply the matrix rank of A_1. It has also been completely solved if $p=2$ for algebraically closed fields by the Kronecker theory of matrix pencils (see [3] which also summarizes Kronecker's results). For $p \geqq 3$ there is no corresponding theory and it seems unlikely that the tensor rank can in general be described in terms of simpler invariants of the tensor.

When working with some unknown $m \times n \times p$ tensor defined by $m \times n$ matrices $A_1,\cdots,A_p$ it is common to assume that the tensor is nondegenerate (3-nondegenerate in the terminology of [4], i.e., the matrices are linearly independent). If the tensor was degenerate it would reduce to one of size smaller than $m \times n \times p$. The assumption of nondegeneracy obviously implies that $p \leqq mn$; moreover, if $p=mn$, the tensor rank is mn. The observations in this paper centre on nondegenerate tensors over algebraically closed fields which have p just less than mn. The case $p=mn-1$ has already been treated in [1], where it was shown that every $m \times n \times (mn-1)$ tensor has rank at most $mn-1$ (and hence the nondegenerate tensors have rank precisely $mn-1$). It then follows that a nondegenerate $m \times n \times (mn-2)$ tensor has rank $mn-1$ or $mn-2$. We shall classify those $m \times n \times (mn-2)$ tensors of rank $mn-1$. Somewhat surprisingly it turns out that, among the infinitely many different tensors, only one of them has this rank. As a consequence we shall prove that every $m \times n \times (mn-3)$ tensor has rank at most $mn-2$ (this result was given in [1] but the proof, which at that time required very extensive calculations, was omitted). Both of these results depend on the following proposition for which we require two definitions:

1. A space of $m \times n$ matrices is said to be perfect if it is generated (as a vector space) by rank 1 matrices. (Notice that the above remarks imply that spaces of dimension mn or $mn-1$ are necessarily perfect.)

2. Two spaces $\mathscr{X}$, $\mathscr{Y}$ of $m \times n$ matrices are said to be equivalent if there exist nonsingular $m \times m$, $n \times n$ matrices P, Q such that

$$\mathscr{Y} = P\mathscr{X}Q = \{PXQ : X \in \mathscr{X}\}.$$

* Received by the editors August 25, 1981.

† Department of Computing Mathematics, University College, Cardiff, United Kingdom.

PROPOSITION. *If $\mathscr{X}$ is a vector space of $m \times n$ matrices of dimension $mn-2$, then $\mathscr{X}$ is perfect unless it is equivalent to the space of all matrices (x_{ij}) for which $x_{11}+x_{22}=0$ and $x_{12}=0$.*

The lemmas which lead up to the proof of this proposition all use the same general technique and notation. Let ϕ be the bilinear mapping on the space of all $m \times n$ matrices defined by

$$\phi(X, Y) = \text{trace}\,(XY')$$

(where Y' denotes the transpose of Y). With respect to ϕ every k-dimensional space $\mathscr{X}$ of $m \times n$ matrices has an annihilator space $\mathscr{X}^*$ of dimension $mn-k$, namely

$$\mathscr{X}^* = \{Y : \phi(X, Y) = 0\}.$$

Of course $\mathscr{X}^{**} = \mathscr{X}$. Also, a routine check shows that, if P and Q are nonsingular, the annihilator of $P\mathscr{X}Q$ is $(P')^{-1}\mathscr{X}^*(Q')^{-1}$ and hence equivalent spaces have equivalent annihilators. We eventually exploit this fact to prove the proposition by taking $\mathscr{X}^*$ to be defined by a pencil in Kronecker canonical form. In preparation for this our first three technical lemmas consider special cases for $\mathscr{X}^*$; these special cases are the building blocks of a general Kronecker canonical form. Note that the annihilator of the exceptional space in the proposition is generated by $E_{11}+E_{22}$ and E_{12} (where, in general, E_{ij} is the matrix with a 1 in the (i, j) position and zeros elsewhere).

LEMMA 1. *Let $\mathscr{A}$ be the (r^2+r-2)-dimensional space of $r \times (r+1)$ matrices whose annihilator is defined by the matrix pencil*

$$L_r = \alpha X + \beta Y = \begin{bmatrix} \alpha & \beta & & & 0 & \\ & \alpha & \beta & & & \\ & & & \ddots & \ddots & \\ & 0 & & & \alpha & \beta \end{bmatrix}.$$

Then $\mathscr{A}$ is perfect.

Proof. By definition $\mathscr{A}$ is the set of all matrices (a_{ij}) for which $\sum_{i=1}^r a_{ii} = \sum_{i=1}^r a_{i,i+1} = 0$. An obvious calculation verifies that the following r^2+r-2 matrices are all rank 1, linearly independent and satisfy the conditions for membership of $\mathscr{A}$:

$$\begin{aligned} &\{E_{ij} : i \neq j, i+1 \neq j\} \\ &\quad \cup \{E_{ii}+E_{i,i+1}+E_{i,i+2}-E_{i+1,i}-E_{i+1,i+1}-E_{i+1,i+2} : i = 1,2,\cdots, r-1\} \\ &\quad \cup \{E_{ii}-E_{i,i+1}+E_{i,i+2}+E_{i+1,i}-E_{i+1,i+1}+E_{i+1,i+2} : i = 1,2,\cdots, r-1\}. \end{aligned}$$

Similarly the transposed space $\mathscr{A}'$ of $(r+1) \times r$ matrices is also perfect.

LEMMA 2. *Let $\mathscr{A}$ be the space of $r \times r$ matrices whose annihilator is defined by the matrix pencil*

$$J_r(\lambda) = \alpha X + \beta Y = \begin{bmatrix} \alpha+\beta\lambda & \beta & & 0 \\ & \alpha+\beta\lambda & \beta \ \ddots & \\ & & \ddots & \beta \\ & 0 & & \alpha+\beta\lambda \end{bmatrix}.$$

Then, if $r \neq 2$, $\mathscr{A}$ is perfect.

Proof. The notation is intended to include the case $r=1$ when the lemma holds for trivial reasons. If $r=2$ $\mathscr{A}$ is not perfect (and this is why the proposition has an exceptional space). We now take $r>2$. In this case $\mathscr{A}$ is the set of all (a_{ij}) for which $\sum_{i=1}^r a_{ii} = \sum_{i=1}^{r-1} a_{i,i+1} = 0$ and $\mathscr{A}$ has dimension r^2-2. The following set is a basis of rank 1 matrices:

$$\{E_{ij}: i \neq j, i+1 \neq j\}$$
$$\cup \{E_{ii}+E_{i,i+1}+E_{i,i+2}-E_{i+1,i}-E_{i+1,i+1}-E_{i+1,i+2}: i=1,2,\cdots,r-2\}$$
$$\cup \{E_{ii}-E_{i,i+1}+E_{i,i+2}+E_{i+1,i}-E_{i+1,i+1}+E_{i+1,i+2}: i=1,2,\cdots,r-2\}$$
$$\cup \{E_{r-2,r-1}-E_{r-2,r}+E_{r-1,r-1}-E_{r-1,r}+E_{r,r-1}-E_{rr}\}.$$

LEMMA 3. *Let $\mathscr{A}$ be the space of 4×4 matrices whose annihilator is defined by the matrix pencil*

$$J_2(\lambda)\oplus J_2(\mu)=\alpha X+\beta Y=\begin{bmatrix} \alpha+\beta\lambda & \beta & 0 & 0 \\ 0 & \alpha+\beta\lambda & 0 & 0 \\ 0 & 0 & \alpha+\beta\mu & \beta \\ 0 & 0 & 0 & \alpha+\beta\mu \end{bmatrix}.$$

Then $\mathscr{A}$ is perfect.

Proof. $\mathscr{A}$ is the set of all 4×4 matrices which satisfy $\sum_{i=1}^{4} a_{ii} = \lambda(a_{11}+a_{22})+\mu(a_{33}+a_{44})+a_{12}+a_{34}=0$. The following set is a basis of rank 1 matrices:

$$\{E_{ij}: i\neq j, (i,j)\neq (1,2) \text{ or } (3,4)\}$$

$$\cup \left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \mu-\lambda & 0 & 0 & \mu-\lambda \\ -1 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \mu-\lambda & 0 & \mu-\lambda \\ 0 & -1 & 0 & -1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \right\}.$$

LEMMA 4. *Let $\mathscr{X}$ be a space of $m\times n$ matrices of dimension $mn-1$ or $mn-2$ all partitioned in some fixed way as $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ where A is an $r\times s$ matrix. Suppose that $\mathscr{X}$ contains all $m\times n$ matrices of the form $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ and suppose also that the subspace $\mathscr{A}=\{\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]\in\mathscr{X}\}$ is perfect but not of dimension rs. Then $\mathscr{X}$ itself is perfect.*

Proof. Let

$$\mathscr{Y}=\{\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]\} \quad \text{(a subspace of } \mathscr{X} \text{ by hypothesis)},$$
$$\mathscr{D}=\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right]\in\mathscr{X}\},$$
$$\mathscr{D}_1=\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right]: \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\in\mathscr{X} \text{ for some } \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\}.$$

Then $\mathscr{D}\subseteq\mathscr{D}_1$ and both have dimension at least $(m-r)(n-s)-2$. In fact $\mathscr{D}_1$ is perfect. For if this were not the case $\mathscr{D}_1$ would have dimension $(m-r)(n-s)-2$ and so would be equal to $\mathscr{D}$. But then $\mathscr{D}$ would be the image of $\mathscr{X}$ under the linear mapping $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\to\left[\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right]$ and $\mathscr{A}\oplus\mathscr{Y}$ would be its kernel from which we would obtain $\mathscr{X}=\mathscr{A}\oplus\mathscr{Y}\oplus\mathscr{D}$; it would follow that $\mathscr{A}$ had dimension rs contradicting the hypotheses.

Consequently we may take a basis of $\mathscr{X}$ consisting of all $E_{ij}\in\mathscr{Y}$, a basis of rank 1 matrices of $\mathscr{A}$ (since $\mathscr{A}$ is perfect), together with certain matrices of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$, where each matrix D has rank 1. These latter matrices $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ may not have rank 1 and to complete the proof we show that by adding a suitable matrix in $\mathscr{A}\oplus\mathscr{Y}$ to every such matrix we can obtain a new basis of $\mathscr{X}$ consisting entirely of rank 1 matrices.

Consider then any one of these basis matrices $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathscr{A}$ we may replace $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ in the basis by $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right] - \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] \notin \mathscr{A}$ then $\langle \mathscr{A}, \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] \rangle$ has dimension rs or $rs-1$ and so is perfect, generated say by the basis of $\mathscr{A}$ and one further rank 1 matrix $\left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then we have $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] = \theta \left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathscr{A}$, $\theta \neq 0$, and hence $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right] = \theta \left[\begin{smallmatrix} T & 0 \\ 0 & D \end{smallmatrix}\right] + \left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since θT and D each have rank 1 we may let $\theta T = u_1 v_1$, $D = u_2 v_2$ for suitable column vectors u_1, u_2 and row vectors v_1, v_2. The matrix $M = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right][v_1 v_2]$ has rank 1 and

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = M - \begin{bmatrix} 0 & u_1 v_2 \\ u_2 v_1 & 0 \end{bmatrix} + \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ can be replaced in the basis by M. Since this can be done for each of the basis matrices $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ the proof is complete.

Proof of proposition. We begin by replacing $\mathscr{X}$ by an equivalent space chosen so that $\mathscr{X}^*$ is spanned by two matrices which define a pencil in Kronecker canonical form [3] without any infinite elementary divisors. By hypothesis the pencil decomposition is not of the form

$$J_2(\lambda) \oplus 0 = \alpha X + \beta Y = \begin{bmatrix} \alpha + \beta\lambda & \beta & 0 \\ & \alpha + \beta\lambda & \\ 0 & & 0 \end{bmatrix}$$

and therefore we may take it to be

$$\alpha \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} + \beta \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix},$$

where the subpencil $\alpha X_1 + \beta Y_1$ is one of the pencils which figure in Lemmas 1,2,3. By definition of $\mathscr{X}^*$ the space $\mathscr{X}$ contains all matrices of the form $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ and the subspace $\mathscr{A}$ of all $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ in $\mathscr{H}$ has, as its annihilator, the space defined by the pencil $\alpha X_1 + \beta Y_1$. $\mathscr{X}$ therefore satisfies the conditions of Lemma 4 and consequently is perfect.

An immediate consequence of the proposition is that any nondegenerate $m \times n \times (mn-2)$ tensor of rank $mn-1$ is equivalent to the one defined by the $mn-2$ matrices $\{E_{11}+E_{22}\} \cup \{E_{ij}: (i,j) \neq (1,1), (1,2), (2,2)\}$. In fact the assumption of nondegeneracy can be dropped. For suppose $A_1, \cdots, A_{mn-2}$ are linearly dependent matrices defining some degenerate $m \times n \times (mn-2)$ tensor. Then the annihilator of the space $\langle A_1, \cdots, A_{mn-2} \rangle$ is of dimension more than 2, and it is easy to prove that it contains a two-dimensional subspace not equivalent to the one generated by $E_{11}+E_{22}$ and E_{12}. Hence $\langle A_1, \cdots, A_{mn-2} \rangle$ is contained in a perfect space of dimension $mn-2$ and so its tensor cannot have rank more than $mn-2$. In particular, every $m \times n \times (mn-3)$ tensor has rank at most $mn-2$.

Finally we note that the assumption about algebraic closure cannot in general be omitted. For example, with $m = n = 2$, the two-dimensional space of all matrices $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ contains no real rank 1 matrix so, over the real field, is neither perfect nor equivalent to the one generated by $E_{11}+E_{22}$ and E_{12}.

Our results can be summarized as follows: For $m \times n \times (mn-2)$ tensors which do not reduce to smaller tensors we have completely solved the ranking problem in the case of algebraically closed fields. Hitherto the only other nontrivial class of tensors for which the ranking problem had been solved was the case of $m \times n \times 2$ tensors [3]. Both solutions rely on Kronecker's theory of pencils but apply it in very different ways.

REFERENCES

[1] M. D. ATKINSON AND N. M. STEPHENS, *On the maximal multiplicative complexity of a family of bilinear forms*, Linear Algebra and Appl., 27 (1979), pp. 1–8.

[2] R. W. BROCKETT AND D. DOBKIN, *On the optimal evaluation of a set of bilinear forms*, Linear Algebra and Appl., 19 (1978), pp. 207–235.

[3] J. JA'JA', *Optimal evaluation of pairs of bilinear forms*, this Journal, 8 (1979), pp. 443–462.

[4] J. KRUSKAL, *Three-way arrays: rank and uniqueness of trilinear decompositions, with applications to arithmetic complexity and statistics*, Linear Algebra and Appl., 18 (1977), pp. 95–138.