Permutations of a multiset avoiding permutations of length 3

M. H. Albert

Department of Computer Science

R. E. L. Aldred

Department of Mathematics and Statistics

M. D. Atkinson

Department of Computer Science

C. Handley

Department of Computer Science

D. Holton

Department of Mathematics and Statistics

University of Otago

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Proof copies to:

Michael Albert

Department of Computer Science

University of Otago

P.O. Box 56, Dunedin

New Zealand.

Abstract

We consider permutations of a multiset which do not contain certain ordered patterns of length 3. For each possible set of patterns we provide a structural description of the permutations avoiding those patterns, and in many cases a complete enumeration of such permutations according to the underlying multiset.

1 Background

Let $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_n)$ be two sequences of numbers. The sequence α is said to be *contained in β as a pattern* (or be *involved in β*) if there is a subsequence $b_{i_1}, \ldots, b_{i_m}$ (with $i_1 < i_2 < \cdots < i_m$) of β which is order isomorphic to α; in other words, $a_r \leq a_s$ if and only if $b_{i_r} \leq b_{i_s}$. If β does not contain α we say that β *avoids* α. This notion has surfaced many times over the last few years in both combinatorial and computing settings, see [3, 4, 5, 9, 13, 14, 15, 16] for example, where attention has focused on the case that the sequences in question have distinct elements (in which case they are generally taken to be permutations of $\{1, 2, \ldots\}$). In all those works a central problem has been to determine the number of permutations of each length that avoid all patterns from some given set.

In this paper we take a more general point of view by considering *permutations of a multiset* $1^{a_1}2^{a_2}\ldots k^{a_k}$. Such permutations are sequences of length $a_1 + a_2 + \cdots + a_k$ which contain a_i occurrences of i for each $1 \leq i \leq k$. Hitherto this generalisation has hardly been considered; indeed, the only substantial work that we know of is the thesis of Alex Burstein [6] in which he mainly considers words in some finite alphabet that avoid sets of patterns. Yet the generalisation to multisets is entirely natural. For example, one of the chief sources of pattern-avoiding sequences are those sequences which a particular data structure is capable of sorting (see [2, 7, 8, 15]); it seems sensible to allow the input to such structures to be arbitrary sequences. Another reason for generalising to multisets is that it allows techniques that are unavailable in the permutation case (in other words, we come across techniques that are not just generalisations of permutation techniques).

Burstein's work is a follow-up to the paper [10] in which the authors considered permutations that avoid the various sets of permutations of length 3. For each such set they gave a formula for the number of permutations of length n. Since there are 6 permutations of length 3 there are at most 2^6 different sets to consider but, in fact, they did not need to look at nearly as many as this; they first grouped them into symmetry classes (under 8 natural symmetries) and handled each symmetry class in turn. Burstein carried out a similar analysis and counted words of length n in a k letter alphabet.

The aim of this paper is to do the same with permutations of an arbitrary multiset $1^{a_1}2^{a_2}\ldots k^{a_k}$. So, we define $S(a_1,\ldots,a_k)$ to be the number of permutations of $1^{a_1}2^{a_2}\ldots k^{a_k}$ which avoid every permutation in the set S. We would like to find a formula for $S(a_1,\ldots,a_k)$ or at least a generating function, or a recurrence, when S consists of length 3 permutations only.

The generating functions we shall work with have the form

$$\sum_{0\le a_1,a_2,\cdots,a_k} S(a_1,\ldots,a_k)\,x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}.$$

In some cases even the generating function is out of reach, and we shall have to be content with a recurrence relation for $S(a_1,\ldots,a_k)$ that enables it to be computed numerically.

2 Symmetry classes

Suppose S is a set of permutations in S_n, thought of as sequences of length n. We let S^R denote the set of reversals of the permutations of S and let $\bar{S}$ denote the $(n+1)$-complement of S (in which every symbol i gets replaced by $n+1-i$). Then, by considering the effect of reversal and $(k+1)$-complement on the class $S(a_1, a_2, \ldots, a_k)$ it is easily seen that

$$S(a_1, \ldots, a_k) = S^R(a_1, \ldots, a_k)$$

and

$$S(a_1, \ldots, a_k) = \bar{S}(a_k, \ldots, a_1).$$

So if we have solved the enumeration problem for the set S and all possible $a_1, a_2, \ldots$ we shall have a solution for S^R and $\bar{S}$.

In the case that S is a set of permutations of length 3 the operations of reversal and complement break the set of possibilities for S into classes with representatives as follows:

Class name	Permutations	Enumeration
A_1	$\{123\}$	Catalan numbers
A_2	$\{132\}$	Catalan numbers
B_1	$\{123, 132\}$	2^{n-1}
B_2	$\{123, 231\}$	$\binom{n}{2} + 1$
B_3	$\{123, 321\}$	zero for $n > 4$
B_4	$\{132, 213\}$	2^{n-1}
B_5	$\{132, 231\}$	2^{n-1}
B_6	$\{132, 312\}$	2^{n-1}
C_1	$\{123, 132, 213\}$	Fibonacci numbers
C_2	$\{123, 132, 231\}$	n
C_3	$\{123, 132, 312\}$	n
C_4	$\{123, 132, 321\}$	zero for $n > 4$
C_5	$\{123, 231, 312\}$	n
C_6	$\{132, 213, 231\}$	n

This table also gives the enumeration results for the classes of ordinary permutations of length n that avoid each permutation of the class. It omits those sets S with $|S| \geq 4$; we shall summarise the (rather easy) results in those cases towards the end of the paper.

From now on let N denote the size $a_1 + \ldots + a_k$ of the multiset $\mathcal{S} = 1^{a_1}2^{a_2}\ldots k^{a_k}$. We now consider the various possibilities for S in turn. In our analysis we shall usually have the tacit assumption that $a_i > 0$ for each i (on the grounds that, if any symbol i does not occur, we can rename the symbols and reduce the value of k). We depart from this assumption in those cases where we wish to consider a generating function, since the algebraic relationships become somewhat less complicated when we allow some $a_i = 0$. Also, we often tacitly assume that $k \geq 3$ since we always have $S(a_1) = 1$ and $S(a_1, a_2) = \binom{a_1+a_2}{a_1}$.

3 Cases of type A

Lemma 1 *$A_1(a_1) = 1$ and for $k \geq 2$,*

$$A_1(a_1, \ldots, a_k) = \begin{cases} A_1(a_1 + a_2, a_3, \ldots, a_k) + \sum_{q=0}^{a_1-1} A_1(a_1 - q, a_2 - 1, \ldots, a_k) & \text{for } a_2 \geq 1 \\ A_1(a_1, a_3, .., a_k) & \text{for } a_2 = 0. \end{cases}$$

PROOF: Assume $a_2 \geq 1$ since the result for $a_2 = 0$ follows by simple relabelling. Let σ be any permutation of $\mathcal{S}$ avoiding 123 in which all the 2's in σ come before all the 1's. Let σ' be the sequence obtained from σ by replacing all the 1's by 2's. Then σ' is a 123-avoiding permutation of $2^{a_1+a_2} \ldots k^{a_k}$. Conversely, for any 123-avoiding permutation σ' of $2^{a_1+a_2} \ldots k^{a_k}$ we can construct a sequence σ of the type considered above by replacing the last a_1 2's in σ' by 1's. Hence the total number of elements of $A_1(a_1, a_2, \ldots, a_k)$ with all the 2's before all the 1's is $A_1(a_1 + a_2, a_3, \ldots, a_k)$.

Next consider a permutation σ of $\mathcal{S}$ avoiding 123 and containing at least one occurrence of a 1 before a 2. By considering the last occurrence of 2 in σ and the last 1 preceding this 2 we obtain a unique decomposition $\sigma = \sigma_1 1 \sigma_2 2 \alpha$

where α contains no 2's and σ_2 contains no 1's. But then, because σ contains no 123 pattern, α has the form $\alpha = 1^q$ for some $q \geq 0$.

Let the sequence $\hat{\sigma} = \sigma_1 1 \sigma_2$ be obtained by removing the uniquely determined final substring 21^q from σ. Clearly $\hat{\sigma}$ has no 123 pattern either and so is one of the $A_1(a_1 - q, a_2 - 1, a_3, \ldots, a_k)$ 123-avoiding permutations of the sequence $1^{a_1-q}2^{a_2-1}\ldots k^{a_k}$.

Conversely, from any one of these latter permutations $\hat{\sigma}$ we can obtain a sequence of $A_1(a_1, a_2, a_3, \ldots, a_k)$ that has at least one occurrence of a 1 before a 2 merely by appending 21^q to $\hat{\sigma}$ (since this cannot introduce a 123 pattern). Therefore the total number of sequences in $A_1(a_1, a_2, a_3, \ldots, a_k)$ which do not have all their 1's before all their 2's is

$$\sum_{q=0}^{a_1-1} A_1(a_1 - q, a_2 - 1, \ldots, a_k)$$

which proves the result. ■

The recurrence in this lemma arises from a division into two cases the first of which is handled by an argument that only applies in the context of multisets. Thus, in one respect at least, the multiset problems are easier than the permutation problems. We shall see several examples of this in the following sections.

In [1] it was proved that $A_2(a_1, a_2, a_3, \ldots, a_k)$ satisfies exactly the same recurrence as in the previous lemma, and therefore it follows that

$$A_1(a_1, a_2, a_3, \ldots, a_k) = A_2(a_1, a_2, a_3, \ldots, a_k).$$

Also in [1] an explicit formula was given for the multi-variate generating function

$$\sum_{a_1,a_2,\dots,a_k} A_2(a_1,\dots,a_k)x_1^{a_1}x_2^{a_2}\dots x_k^{a_k}.$$

It was observed to be a symmetric function, and hence that $A_2(a_1, a_2, \dots, a_k)$ must be symmetric in its arguments. We now give a direct proof of a more general result.

Theorem 2 *Let r be fixed and let $e(a_1, \dots, a_k)$ be the number of permutations of the multiset $1^{a_1}2^{a_2}\dots k^{a_k}$ that avoid $1, 2, \dots, r+1$. Then $e(a_1, \dots, a_k)$ is a symmetric function.*

Proof: We recall the definition of the Schur function $s_\lambda(x)$ where $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\lambda_1 \geq \dots \geq \lambda_t$ is a partition of n and $x = (x_1, x_2, \dots)$ is a set of indeterminates. We have

$$s_\lambda = \sum_T x^T$$

where the sum is over all semistandard Young tableaux T of shape λ and x^T denotes the type enumerator $x_1^{a_1}x_2^{a_2}\dots$ of T (in other words, the entries of T comprise the multiset $1^{a_1}2^{a_2}\dots$). By definition, the coefficients of s_λ tell us how many semistandard Young tableau of shape λ there are for each possible multiset of entries.

Next, let f_λ denote the number of standard Young tableaux on the set $1, 2, \dots, n = \sum \lambda_i$ of shape λ. Then $f_\lambda s_\lambda$ enumerates the pairs (S, T) of Young tableaux of shape λ according to each possible multiset of entries for T and where S has entries $1, 2, \dots, n$.

Now put $E_r(x) = \sum_\lambda s_\lambda f_\lambda$ where the sum is over partitions with maximum part at most r. Since the Schur functions are symmetric (Theorem 7.10.2 of [11]) so also is $E_r(x)$.

However, applying the Robinson-Schensted-Knuth correspondence to two-line arrays with first line $1, 2, \ldots, n$ and second line a permutation of $1^{a_1} 2^{a_2} \ldots k^{a_k}$ gives a one-to-one correspondence between permutations of $1^{a_1} 2^{a_2} \ldots k^{a_k}$ that have no increasing subsequence of length $r + 1$ and the Young tableau pairs (S, T) introduced above. Consequently the coefficients of $E_r(x)$ enumerate these permutations for each multiset, and the theorem follows. ∎

4 Cases of type B

4.1 $B_1 = \{123, 132\}$

Let σ be a permutation of $\mathcal{S}$ that avoids the elements of B_1. Suppose first that all the 2's of σ precede all the 1's. By a similar argument to that given in the proof of Lemma 1 there are $B_1(a_1 + a_2, a_3, \ldots, a_k)$ possibilities for σ. In the contrary case (at least one 1 followed by a 2 somewhere) we may write σ as $\alpha 1 \beta$ with no 1's in α and at least one 2 in β. Then β must consist of 1's and 2's alone and α can be any permutation avoiding the permutations of B_1. This case contributes $\sum_{u=1}^{a_2} \binom{a_1+u-1}{u} B_1(a_2 - u, a_3, \ldots)$ permutations so

$$B_1(a_1, a_2, \ldots, a_k) = B_1(a_1 + a_2, a_3, \ldots, a_k) + \sum_{u=1}^{a_2} \binom{a_1 + u - 1}{u} B_1(a_2 - u, a_3, \ldots, a_k). \qquad (1)$$

If $k = 2$ then, of course, $B_1(a_1, a_2) = \binom{a_1+a_2}{a_1}$. When $k = 3$ we may simplify the binomial summation to obtain:

$$B_1(a_1, a_2, a_3) = \binom{a_1 + a_2 + a_3}{a_3} - \binom{a_2 + a_3}{a_3} + \binom{a_1 + a_2 + a_3}{a_2}.$$

However, such a simple form does not exist for $k \geq 4$.

The recurrence (1) allows $B_1(a_1, a_2, \ldots, a_k)$ to be computed in stages. Suppose, for some i in $k, k-1, \ldots, 2$, we have found $B_1(a_1+a_2+\ldots+a_i, a_{i+1}, \ldots, a_k)$ and all $B_1(j, a_{i+1}, \ldots, a_k)$ with $1 \leq j \leq a_2+\ldots+a_i$. We can then use the recurrence to compute the analogous quantities in which i is replaced by $i-1$. The cost of each such stage is $O(a_i(a_2 + \ldots + a_{i-1}))$ and the total cost is therefore $O(N^2)$.

Since the recurrence for B_1 still holds when we allow any $a_i = 0$ it can be viewed as a relationship concerning the generating functions b_k for B_1. Recall that:

$$b_k(x_1, x_2, \ldots, x_k) = \sum_{0 \leq a_1, a_2, \ldots, a_k} B_1(a_1, \ldots, a_k)\, x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$$

but for notational simplicity we may drop the subscript k which can be inferred from the number of variables in the list of arguments.

The first term of the recurrence is the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in

$$\frac{x_1 b(x_1, x_3, \ldots, x_k) - x_2 b(x_2, x_3, \ldots, x_k)}{x_1 - x_2}$$

as can easily be seen by setting $t = a_1 + a_2$ and considering the expression above as:

$$\sum_{0 \leq t, a_3, \ldots, a_k} B(t, a_3, \ldots, a_k) \frac{x_1^{t+1} - x_2^{t+1}}{x_1 - x_2}.$$

Now, note that:

$$\binom{a_1+u-1}{u} x_2^u = \binom{-a_1}{u} (-x_2)^u$$

so the summation term in the recurrence is precisely the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in:

$$\left(\sum_{a=0}^{\infty} \sum_{u=1}^{\infty} \binom{-a}{u} x_1^a (-x_2)^u \right) b(x_2, x_3, \ldots, x_k).$$

If the summation on u were from 0, the inner sum would be $x_1^a/(1-x_2)^a$, and the whole summation would equal

$$\frac{1-x_2}{1-x_1-x_2} b(x_2, x_3, \ldots, x_k).$$

However, the extra terms added for $u = 0$ contribute precisely

$$\frac{1}{1-x_1} b(x_2, x_3, \ldots, x_k)$$

to the actual sum. Putting all this information together we obtain:

$$\begin{aligned} b(x_1, x_2, \ldots x_k) &= \frac{x_1}{x_1 - x_2} b(x_1, x_3, \ldots, x_k) \\ &+ \left(\frac{-x_2}{x_1 - x_2} + \frac{1-x_2}{1-x_1-x_2} - \frac{1}{1-x_1} \right) b(x_2, x_3, \ldots, x_k). \end{aligned}$$

4.2 $B_2 = \{123, 231\}$

Let σ be a permutation of $\mathcal{S}$ that avoids the permutations of B_2. Consider the positions of the k's and $(k-1)$'s in σ. Suppose first that all the k's precede the $k-1$'s. In this case we can replace the k's by $k-1$'s without creating a forbidden pattern. So there are $B_2(a_1, a_2, \ldots, a_{k-2}, a_{k-1}+a_k)$ sequences of this type. Now suppose that some $k-1$ precedes a k. Take the first $k-1$ and the final k. All the symbols that are less than $k-1$ must appear between this pair, and they must occur in descending order. So we have:

$$\sigma = k^a(k-1)^b(k-2)^{a_{k-2}}\cdots 2^{a_2}1^{a_1}k^c(k-1)^d.$$

Since $1 \le b \le a_{k-1}$ and $1 \le c \le a_k$ there are $a_{k-1}a_k$ sequences of this type. Thus

$$B_2(a_1, a_2, \ldots, a_k) = B_2(a_1, a_2, \ldots, a_{k-2}, a_{k-1}+a_k) + a_{k-1}a_k.$$

Moreover $B_2(a_1, a_2) = \binom{a_1+a_2}{a_1}$ and so we get inductively that:

$$B_2(a_1, a_2, \ldots, a_k) = \binom{a_1+a_2+\cdots+a_k}{a_1} + \sum_{2\le i<j\le k} a_i a_j.$$

4.3 $B_3 = \{123, 321\}$

Let σ be a permutation of $\mathcal{S}$ that avoids the permutations of B_3. It is a special case of the Erdős-Szekeres lemma that at most 4 distinct symbols can occur in σ so the only cases of interest are when there are three or four distinct symbols. First consider the case that only symbols $1, 2, 3$ occur in σ. Then for some β consisting only of 1's and 3's:

$$\sigma = 2^u\beta 2^{a_2-u}. \tag{2}$$

for some $0 \le u \le a_2$ (since, once a 1 or a 3 occurs, all the symbols of the complementary type must occur before the next 2, but then all the 1's and 3's must occur in this block). So the number of sequences of this type is:

$$B_3(a_1, a_2, a_3) = (a_2+1)\binom{a_1+a_3}{a_1}.$$

Now suppose that σ contains $1,2,3,4$. If somewhere a 3 follows a 2 then no 4 may follow that 3, nor may a 4 precede the 2, since in that case there would be no legal place for any 1. So all the 4's, and symmetrically all the 1's must lie between all the 2's and all the 3's. On the other hand, any pattern of 1's and 4's between the 2's and 3's is allowed. A similar argument also holds if, in σ, a 2 follows a 3. Thus:

$$B_3(a_1, a_2, a_3, a_4) = 2\binom{a_1 + a_4}{a_1}.$$

4.4 $B_4 = \{132, 213\}$

Let σ be a permutation of $\mathcal{S}$ avoiding the permutations of B_4. We divide the analysis into two cases. The first is that all the 1 symbols come before all the 2 symbols. Then because 132 is avoided there are no other symbols between the 1 and 2 symbols and σ is $\alpha 1^{a_1} 2^{a_2} \beta$ with neither α nor β containing any symbols $1, 2$. There are $B_4(1, a_3, ..., a_k)$ sequences of this type (to obtain a correspondence, replace the central block by a single 2, then subtract 1 from every symbol).

The second case is that some 1 symbol is to the right of some 2 symbol. Since 213 is avoided there can only be symbols 2 (if any) to the right of the rightmost 1 symbol. If there is no final symbol 2 so that σ ends in 1 we can count these as $B_4(a_1 - 1, a_2, \ldots, a_k)$. But if there is a 2 symbol at the end then σ must end with a block of mixed 1 and 2 symbols that includes *all* the 1 symbols. There are $\binom{a_1+j-2}{j-1} B_4(a_2 - j, a_3, ..., a_k)$ such sequences in which j of the 2's occur in this block. In all we have

$$\begin{aligned}B_4(a_1, a_2, \ldots, a_k) &= B_4(1, a_3, ..., a_k) + B_4(a_1 - 1, a_2, \ldots, a_k) \\ &+ \sum_{j=1}^{a_2} \binom{a_1 + j - 2}{j - 1} B_4(a_2 - j, a_3, ..., a_k) - B_4(a_3, \ldots, a_k)\end{aligned}$$

where the subtractive final term is because the sequences of the form $\alpha 1^{a_1} 2^{a_2}$ have been counted twice.

A simplification can be made by replacing a_1 by r and writing this equation as

$$B_4(r, a_2, \ldots, a_k) - B_4(r-1, a_2, \ldots, a_k) =$$
$$B_4(1, a_3, \ldots, a_k) - B_4(a_3, \ldots, a_k) + \sum_{j=1}^{a_2} \binom{r + j - 2}{j - 1} B_4(a_2 - j, a_3, \ldots, a_k).$$

Summing from $r = 1$ to a_1 and rearranging terms gives

$$\begin{aligned}B_4(a_1, a_2, \ldots, a_k) &= a_1 B_4(1, a_3, \ldots, a_k) - a_1 B_4(a_3, \ldots, a_k) \\ &+ \sum_{j=0}^{a_2} \binom{a_1 + j - 1}{j} B_4(a_2 - j, a_3, \ldots, a_k)\end{aligned}$$

where a standard binomial identity has been used to simplify the final term.

Notice that the symmetry operation "complement reverse" preserves B_4 and so

$$B_4(a_1, a_2, \ldots, a_k) = B_4(a_k, a_{k-1}, \ldots, a_1).$$

The generating function computation (done for B_1 above and C_1 below) is a little more complicated in this case because of the "1" in the recurrence. We have to define the generating functions $f_k(x_1, \ldots, x_k)$ as before together with the generating function $g_{k-1}(x_2, \ldots, x_k)$ associated with the numbers $B(1, a_2, \ldots, a_k)$.

However, the complications are controllable and give the equations:

$$\begin{aligned} f_k(x_1,\dots,x_k) &= f_{k-1}(x_1,x_3,\dots,x_k) + \frac{1-x_2}{1-x_1-x_2} f_{k-1}(x_2,\dots,x_{k-1}) \\ &\quad - f_{k-2}(x_3,\dots,x_k)\left[\frac{1}{1-x_1} + \frac{x_1x_2}{(1-x_1)^2(1-x_2)}\right] \\ &\quad + g_{k-2}(x_3,\dots,x_k)\frac{x_1x_2}{(1-x_1)^2(1-x_2)}, \end{aligned}$$

$$\begin{aligned} (1-x_2)g_{k-1}(x_2,\dots,x_k) &= g_{k-2}(x_3,\dots,x_k) \\ &\quad - f_{k-2}(x_3,\dots,x_k) + f_{k-1}(x_2,\dots,x_k). \end{aligned}$$

4.5 $B_5 = \{132, 231\}$

Let σ be a permutation of $\mathcal{S}$ avoiding 132 and 231. If $k > 2$ and i and j are distinct symbols less than k no symbol k can occur between them. Therefore $\sigma = k^{u_k}\tau k^{a_k-u_k}$ for some $0 \le u_k \le a_k$. Iterating this argument we find

$$\sigma = k^{u_k}(k-1)^{u_{k-1}}\dots 3^{u_3}\rho 3^{a_3-u_3}\dots(k-1)^{a_{k-1}-u_{k-1}}k^{a_k-u_k} \qquad (3)$$

for some $0 \le u_t \le a_t$ and ρ a permutation of $1^{a_1}2^{a_2}$. Since every such sequence avoids 132 and 231 we have

$$B_5(a_1,\dots,a_k) = \binom{a_1+a_2}{a_1}\prod_{t=3}^{k}(a_t+1).$$

4.6 $B_6 = \{132, 312\}$

Let σ be a permutation of $\mathcal{S}$ that avoids the permutations of B_6. We consider two cases. In the first case all the 2 symbols occur before all the 1 symbols. Here there is a one-to-one correspondence with permutations of $2^{a_1+a_2}3^{a_3}\dots k^{a_k}$ as in the B_1 analysis.

In the second case the last 2 symbol is preceded (somewhere) by a 1 symbol and we write

$$\sigma = \alpha 2 \beta$$

where $2 \notin \beta$ and $\alpha = \alpha_1 1 \alpha_2$. Since σ avoids 312, α_1 contains no symbol larger than 2. Since σ avoids 132, α_2 contains no symbol larger than 2 and also β is non-decreasing in the symbols $3, 4, \ldots, k$. Hence in σ the symbols $2, \ldots, k$ occur in natural increasing order. Recall that $N = |\sigma|$. There are $\binom{N}{a_1}$ permutations of $1^{a_1} 2^{a_3} \ldots k^{a_k}$ in which $2, \ldots, k$ occur in increasing order. Of these $\binom{N-a_2}{a_1}$ have all their 2 symbols at the start (i.e. do not have a symbol 1 before a symbol 2). So this case accounts for $\binom{N}{a_1} - \binom{N-a_2}{a_1}$ sequences. Therefore

$$B_6(a_1, \ldots, a_k) = B_6(a_1 + a_2, a_3, \ldots, a_k) + \binom{N}{a_1} - \binom{N - a_2}{a_1}.$$

Now we iterate this recurrence to obtain

$$B_6(a_1, \ldots, a_k) = \sum_{r=1}^{k-1} \binom{N}{N_r} - \sum_{r=1}^{k-2} \binom{N - a_{r+1}}{N_r}$$

where N_r denotes $\sum_{t=1}^{r} a_t$.

5 Cases of type C

5.1 $C_1 = \{123, 132, 213\}$

We suppose first that σ is a permutation of $\mathcal{S}$ avoiding $123, 132, 213$ and that each of a_{k-1} and a_k is positive. Since σ avoids 123 and 213 the rightmost symbol k cannot be preceded by any two distinct symbols i, j that are less than

k. Furthermore, if there is a symbol less than k and preceding k then that symbol has to be $k-1$ (since σ avoids 132). We may therefore write $\sigma = \alpha k \beta$ where α only contains u symbols $k-1$ for some $0 \le u \le a_{k-1}$ and $a_k - 1$ symbols k. Moreover the symbols in β can be arranged in any permutation avoiding the elements of C_1. Hence

$$C_1(a_1, \dots, a_k) = \sum_{u=0}^{a_{k-1}} \binom{a_k - 1 + u}{u} C_1(a_1, \dots, a_{k-1} - u). \tag{4}$$

This recurrence allows $C_1(a_1, \dots, a_k)$ to be computed in $O(\sum_i a_{i-1} a_i)$ steps. For each $i = 1, \dots, k$ we compute all $C_1(a_1, \dots, a_{i-1}, j)$ for $j = 0, \dots, a_i$ having precomputed all the binomial coefficients.

We now compute a recurrence for the generating function of C_1, as we did for B_1 above. Let c be the required generating function. Then, applying the same technique as we applied to the second part of the recurrence for the class B_1, would suggest that c satisfies:

$$c(x_1, x_2, \dots, x_k) = \frac{1 - x_{k-1}}{1 - x_k - x_{k-1}} c(x_1, x_2, \dots, x_{k-1}).$$

However, this is incorrect as the recurrence given for C_1 fails in the case $a_{k-1} = 0$ as it would imply that:

$$C_1(a_1, \dots, a_{k-2}, 0, a_k) = C(a_1, \dots, a_{k-2})$$

whereas in fact:

$$C_1(a_1, \dots, a_{k-2}, 0, a_k) = C(a_1, \dots, a_{k-2}, a_k).$$

We can deal with this by noting that the terms in the generating function for which $a_{k-1} = 0$ can be isolated by setting $x_{k-1} = 0$. So we need to subtract these erroneous terms and then add terms corresponding to the correct version. This gives:

$$\begin{aligned} c(x_1, x_2, \dots, x_k) &= \frac{1 - x_{k-1}}{1 - x_k - x_{k-1}} c(x_1, x_2, \dots, x_{k-1}) \\ &\quad - \frac{1}{1 - x_k} c(x_1, x_2, \dots, x_{k-2}) + c(x_1, x_2, \dots, x_{k-2}, x_k). \end{aligned}$$

5.2 $C_2 = \{123, 132, 231\}$

To handle the case C_2 we impose the extra condition that sequences of the form (3) should avoid 123. If $\rho = 2^{a_2}1^{a_1}$ then $u_t = a_t$ for all t with perhaps one exception; this accounts for $1 + \sum_{t=3}^{k} a_t$ sequences. On the other hand, if $\rho \neq 2^{a_2}1^{a_1}$ then all $u_t = a_t$ and there are $\binom{a_1+a_2}{a_1} - 1$ such sequences. Hence

$$C_2(a_1, \dots, a_k) = \binom{a_1 + a_2}{a_1} + \sum_{t=3}^{k} a_t.$$

5.3 $C_3 = \{123, 132, 312\}$

Let σ be a permutation of $1^{a_1}2^{a_2}\dots k^{a_k}$ avoiding $123, 132, 312$. Assume that the largest symbol k occurs at least once. Then any two symbols less than k have to occur in non-increasing order. Thus σ is a merge of the sequences k^{a_k} and $(k-1)^{a_{k-1}}\dots 2^{a_2}1^{a_1}$. Therefore

$$C_3(a_1, \dots, a_k) = \binom{a_1 + a_2 + \dots + a_k}{a_k}.$$

5.4 $C_4 = \{123, 132, 321\}$

This case is like case B_3 with the extra condition that permutations σ must avoid 132. Suppose σ only has symbols $1, 2, 3$ and we have equation (2). Then, if there are any 2's at the end of σ, all the 3's must precede the 1's. So the sequence is one of:

$$2^{a_2}\alpha \quad \text{or} \quad 2^{a_2-u}3^{a_3}1^{a_1}2^u \text{ for } u \geq 1,$$

where α is any permutation of $1^{a_1}3^{a_3}$. This gives:

$$C_4(a_1, a_2, a_3) = \binom{a_1 + a_3}{a_1} + a_2$$

again we get:

$$C_4(b_1, b_2) = C_4(b_1, 0, b_2), \quad C_4(b_1) = C_4(b_1, 0, 0).$$

In the case where all four symbols occur, the 2's must not be at the beginning, for a 3 at the end with a 4 in the middle produces the pattern 132. So all the 3's are at the beginning and all the 2's at the end. Then all the 4's must precede all the 1's to avoid 132. So there is only one legitimate sequence in this case:

$$3^{a_3}4^{a_4}1^{a_1}2^{a_2}$$

and

$$C_4(a_1, a_2, a_3, a_4) = 1.$$

5.5 $C_5 = \{123, 231, 312\}$

Suppose that $k > 2$ and let σ be a permutation of $\mathcal{S}$ avoiding $123, 231, 312$ in which at least three different symbols occur, including both 1 and k. Then, either σ is non-increasing or we can find i, j which are consecutive in σ and $i < j$. Such a pair is called an *ascent* of σ. Then, because of the avoided patterns, we must have $i = 1$ and $j = k$. So all ascents of σ have the form $1k$. There cannot be more than one ascent since if $\sigma = \alpha 1k\beta 1k\gamma$ it is easy to see that a symbol equal to neither 1 or k cannot belong to any of α, β, γ. The avoidance of 312 implies that all the symbols preceding the ascent are no larger than any of the symbols following it. Now it follows that σ is a cyclic shift of the unique non-increasing sequence on $\mathcal{S}$. Hence

$$C_5(a_1, \ldots, a_k) = a_1 + a_2 + \cdots + a_k.$$

5.6 $C_6 = \{132, 213, 231\}$

This is a specialisation of the case B_5 and so we consider sequences of the form (3) that avoid 213. We put $\sigma = \alpha\rho\beta$. If $\rho = 1^{a_1}2^{a_2}$ the 213 restriction implies that $a \geq b$ for all $a \in \alpha$ and $b \in \beta$. Such sequences are determined by $|\alpha|$ and so there are $1 + \sum_{t=3}^{k} a_t$ of them. However, if $\rho \neq 1^{a_1}2^{a_2}$ then β is empty and there are $\binom{a_1+a_2}{a_1} - 1$ such sequences. Consequently

$$C_6(a_1, \ldots, a_k) = \binom{a_1 + a_2}{a_1} + \sum_{t=3}^{k} a_t.$$

6 Mopping up

Class name	Basis	Enumeration, $k \geq 3$
D_1	$\{123, 132, 213, 231\}$	$\binom{a_1+a_2}{a_2}$
D_2	$\{123, 132, 231, 312\}$	$a_k + 1$
D_3	$\{132, 213, 231, 312\}$	2
D_4	$\{123, 132, 213, 321\}$	$a_2 + 1$ if $k = 3$, 1 if $k = 4$, 0 if $k > 4$
D_5	$\{123, 231, 132, 321\}$	2 if $k = 3$, 0 if $k > 3$
D_6	$\{123, 213, 231, 321\}$	$\binom{a_1+a_3}{a_1}$ if $k = 3$, 0 if $k > 3$
E_1	$\{123, 132, 213, 231, 312\}$	1
E_2	$\{123, 132, 213, 231, 321\}$	1 if $k = 3$, 0 if $k > 3$

These formulae are all easily obtainable either directly or from the previous analysis with further conditions imposed.

We now have fairly good formulae for all cases except for B_1, B_4, C_1, for which we have recurrences, both for the actual values, and for the associated generating functions.

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