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On Computing the Number of Linear Extensions of a Tree*

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Abstract. An algorithm requiring $O(n^2)$ arithmetic operations for computing the number of linear extensions of a poset whose covering graph is a tree is given.

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Let P be a finite set of n elements on which is defined a partial order $<$. It is an open question [3] whether $L(P)$, the number of linear extensions of the partial order, can be computed in time polynomial in n although polynomial time algorithms are known [2] for some special classes of posets. In this note we consider posets whose covering graph is a tree. It was proved in [1] that, for such posets, $L(P)$ can be computed in $O(n^5)$ arithmetic operations but the algorithm was unwieldy. Here we shall give an $O(n^2)$ algorithm for these posets which admits simple implementation.

If $\alpha \in P$ the *α-spectrum* of P is the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where λ_i is the number of total orderings $(x_1, x_2, \ldots, x_n)$ wherein $x_i = \alpha$ (that is, α has rank i in the total ordering).

LEMMA 1. *Let P, Q be disjoint finite sets of sizes u, v carrying partial orders denoted by $<_P, <_Q$ and let $(\lambda_1, \lambda_2, \ldots, \lambda_u)$ be the α-spectrum of P and $(\mu_1, \mu_2, \ldots, \mu_v)$ the β-spectrum of Q. Consider the partial order relation $<_1$ defined on $R = P \cup Q$ whose covering relations are those of P and Q together with the relation $\alpha <_1 \beta$. Then the r-th member of the spectrum of R is given by*

$$\sum_{i=\max(1,r-v)}^{\min(u,r)} \lambda_i k_{ri} \sum_{j=r-i+1}^{v} \mu_j$$

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where

$$k_{ri} = \binom{r-1}{i-1}\binom{u+v-r}{u-i}.$$

Proof. A linear extension ζ of R in which α has rank r is obtained by merging a linear extension $\xi = (x_1, x_2, \ldots, x_u)$ of P with a linear extension $\psi = (y_1, y_2, \ldots, y_v)$ of Q. If $x_i = \alpha$ for some i in $1 \leqslant i \leqslant u$ then, in order to have α of rank r in ζ, we must have x_i between y_{r-i} and y_{r-i+1} in ζ (see Figure 1) and therefore $0 \leqslant r - i \leqslant v$. Thus i satisfies $\max(1, r - v) \leqslant i \leqslant \min(u, r)$. Since $\alpha <_1 \beta$, β must have rank at least $r - i + 1$ in ψ. On the other hand, given any value of i with $\max(1, r - v) \leqslant i \leqslant \min(u, r)$ there are λ_i possibilities for the linear extension ξ having $x_i = \alpha$ and $\Sigma_{j=r-i+1}^{v} \mu_j$ possibilities for the linear extension ψ. Given possibilities for ξ and ψ there are $\binom{r-1}{i-1}$ ways of merging $(x_1, x_2, \ldots, x_{i-1})$ with $(y_1, y_2, \ldots, y_{r-i})$, and $\binom{u+v-r}{u-i}$ ways of merging $(x_{i+1}, \ldots, x_u)$ with $(y_{r-i+1}, \ldots, y_v)$. Taking all values of i from $\max(1, r-v)$ to $\min(u, r)$ into account we obtain the formula claimed.

Note. There is a similar formula, proved in the same way, for the α-spectrum of the partial order $<_2$ whose covering relations are those of P and Q together with $\beta <_2 \alpha$, namely, its r-th member is

$$\sum_{i=\max(1,r-v)}^{\min(u,r)} \lambda_i k_{ri} \sum_{j=1}^{r-i} \mu_j$$

LEMMA 2. *The calculation of the formulae in Lemma* 1 (*and those of the note following its proof*) *can be carried out using $O(uv)$ arithmetic operations assuming that all the necessary binomial coefficients have been precomputed.*

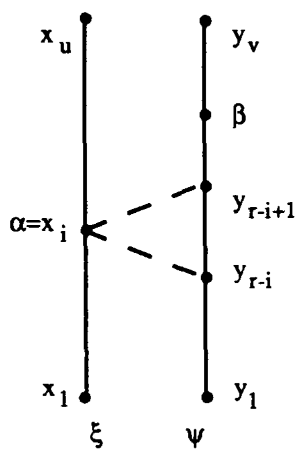

Fig. 1.

Proof. All quantities $\omega_k = \Sigma_{j=k}^{v} \mu_j$, $k = v, v-1, \ldots, 1$, are calculated first in $O(v)$ operations using the equations $\omega_v = \mu_v$ and $\omega_k = \omega_{k+1} + \mu_k$ for $k < v$. Assume first that $u \leqslant v$. The members $\Sigma_{i=\max(1,r-v)}^{\min(u,r)} \lambda_i k_{ri} \omega_{r-i+1}$ of the spectral sequence with $1 \leqslant r \leqslant u$ require a total of $A(1+2+\cdots+u) = \frac{1}{2}Au^2 + O(u)$ operations (A being a constant); for $u < r \leqslant v$ the number of operations is $Au(v-u)$; while for $v < r \leqslant u+v$ the number of operations required is again $A(1+2+\cdots+u) = \frac{1}{2}Au^2 + O(u)$. The total number is therefore $Auv + O(u)$. The case $v \leqslant u$ is analysed in the same way.

PROPOSITION. *Let R be any poset on n elements whose covering graph is a tree and let $\alpha \in R$. Then the α-spectrum (and therefore $L(R)$) can be calculated in $O(n^2)$ arithmetic operations.*

Proof. We begin by computing and storing all the necessary binomial coefficients $\binom{a}{b}$, $0 \leqslant b \leqslant a \leqslant n$, by constructing a Pascal's triangle; this requires $O(n^2)$ operations. Choose any edge of the covering graph incident with α and let β be its other end. When this edge is removed from the graph there remain two components P, Q (containing α, β and of sizes u, v respectively) which are the covering graphs of the posets induced in the underlying subsets. The α-spectrum of P and the β-spectrum of Q may each be calculated by recursion and then the α-spectrum of R may be found using the formulae of Lemma 1.

If $T(n)$ is the number of arithmetic operations required by this algorithm then, by Lemma 2,

$$T(n) \leqslant T(u) + T(v) + Auv, \quad \text{for some constant } A.$$

We can now prove that $T(n) \leqslant \frac{1}{2}An^2$ by induction on n. For, by the inductive hypothesis,

$$T(n) \leqslant \tfrac{1}{2}Au^2 + \tfrac{1}{2}Av^2 + Auv = \tfrac{1}{2}A(u+v)^2 = \tfrac{1}{2}An^2.$$

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