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Subclasses of the separable permutations

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Abstract

The separable permutations are those that can be obtained from the trivial permutation by two operations called direct sum and skew sum. This class of permutations contains the class of stack sortable permutations, Av(231), which are enumerated by the Catalan numbers. We prove that all subclasses of the separable permutations which do not contain Av(231) or a symmetry of this class have rational generating functions. Our principal tools include partial well-order (the lack of an infinite antichain), atomicity (the joint embedding property), and the theory of strongly rational permutation classes which is introduced here for the first time.

1. *Introduction*

The *separable permutations* are those that can be built from the trivial permutation 1 by repeatedly applying two operations, known as *direct sum* (or simply, *sum*) and *skew sum* (or simply, *skew*) which are defined, respectively, on permutations π of length m and σ of length n by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m)+m & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+n & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) & \text{if } m+1 \leqslant i \leqslant m+n. \end{cases}$$

In this introductory section, we recapitulate some known results about the separable permutations and some related sets of permutations. The operations $\oplus$ and $\ominus$ are best understood by considering the plots of the permutations, as in Figure 1.

While the term 'separable permutation' dates only to the work of Bose, Buss, and Lubiw [**6**], these permutations have a long history, going back at least to Avis and Newborn's work on pop stacks [**5**]. Separable permutations are the permutation analog of two other well-studied classes of object: complement-reducible graphs (also called cographs, or simply P_4-free graphs), and series-parallel (or N-free) posets. A folkloric result (which also follows from the characterizations of these analogous classes) characterizes the separable permutations.

Proposition 1.1. *A permutation π is separable if and only if it contains neither* 2413 *nor* 3142.

In Proposition 1.1, we say that the permutation π of length n *contains* the permutation σ of length k (written $\sigma \leqslant \pi$) if π has a subsequence of length k order isomorphic to σ. For example, $\pi = 34918672$ (written in a list, or one-line notation) contains $\sigma = 51342$, as can be seen by considering the subsequence 91672.

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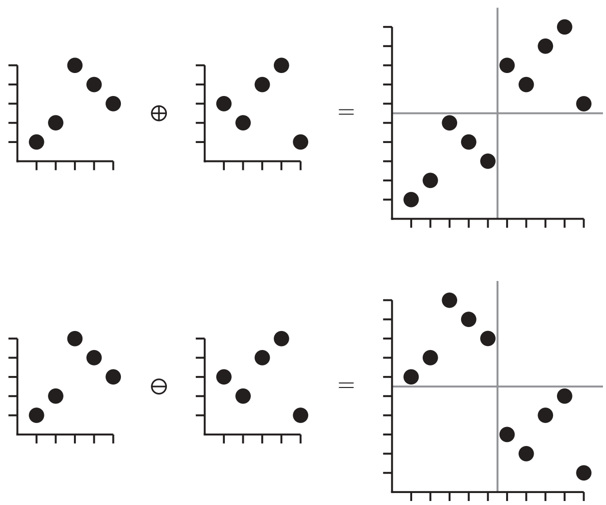

FIGURE 1. *An example of a direct sum and a skew sum.*

Our interest is with sets of permutations which are closed downwards under this containment order, which we call *permutation classes* (or just *classes*). Thus, $\mathcal{C}$ is a class if, for all π in $\mathcal{C}$ and all $\sigma \leqslant \pi$, σ is also in $\mathcal{C}$. One way to specify classes is as closures: if X is any set of permutations, then its *closure* is the permutation class

$$\mathrm{Cl}(X) = \{\sigma : \sigma \leqslant \pi \text{ for some } \pi \in X\}.$$

It is also occasionally useful to specify classes by what they do not contain; for any permutation class $\mathcal{C}$ there is a unique (and possibly infinite) antichain B such that

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi : \pi \not\geqslant \beta \text{ for all } \beta \in B\}.$$

This antichain B is called the *basis* of $\mathcal{C}$, so the basis of the separable class is $\{2413, 3142\}$. Another way to characterize the separable permutations is provided by the next result; in this result, we write $\mathcal{C} \oplus \mathcal{D}$ for the set of permutations of the form $\pi \oplus \sigma$, where π lies in $\mathcal{C}$ and σ lies in $\mathcal{D}$, and extend this definition to $\mathcal{C} \ominus \mathcal{D}$ analogously. (If $\mathcal{C}$ and $\mathcal{D}$ are permutation classes, then so are $\mathcal{C} \oplus \mathcal{D}$ and $\mathcal{C} \ominus \mathcal{D}$.)

PROPOSITION 1.2. *The class of separable permutations is the smallest nonempty class* $\mathcal{C}$ *that satisfies both* $\mathcal{C} \oplus \mathcal{C} \subseteq \mathcal{C}$ *and* $\mathcal{C} \ominus \mathcal{C} \subseteq \mathcal{C}$.

Any class satisfying $\mathcal{C} \oplus \mathcal{C} \subseteq \mathcal{C}$ is called *sum closed*, while any class satisfying $\mathcal{C} \ominus \mathcal{C} \subseteq \mathcal{C}$ is called *skew closed*.

The separable permutations contain a notable subclass Av(231), which Knuth [**11**] showed are precisely the permutations that can be sorted by a stack (a last-in first-out list). A result similar to Proposition 1.2 holds for Av(231) as well.

PROPOSITION 1.3. *The class* Av(231) *is the smallest nonempty class* $\mathcal{C}$ *that satisfies both* $\mathcal{C} \oplus \mathcal{C} \subseteq \mathcal{C}$ *and* $1 \ominus \mathcal{C} \subseteq \mathcal{C}$.

In fact, since both basis elements (2413 and 3142) of the separable class contain every nonmonotone permutation of length 3, all four of the classes Av(132), Av(213), Av(231), and Av(312) are contained in the separable permutations, and each of these has a characterization similar to the one given by Proposition 1.3. These four classes are all symmetric images of one another under the operations of reversal, inverse, and complementation (or compositions of these), each of which preserves the containment order.

For any class $\mathcal{C}$ (or more generally any set of permutations), we denote by $\mathcal{C}_n$ the set of permutations in $\mathcal{C}$ of length n, and say that the *generating function* for $\mathcal{C}$ is $\sum |\mathcal{C}_n| x^n$. Whether this sum includes the empty permutation ($n = 0$) is a matter of taste and convenience, and we generally elect to omit it.

Note that every separable permutation of length at least 2 (and by extension, every permutation in Av(231) of length at least 2) is either *sum decomposable*, meaning that it is equal to $\pi \oplus \sigma$ for two shorter permutations π and σ, or is *skew decomposable*, which is defined analogously. No permutation is both sum and skew decomposable, so the separable permutations may therefore be partitioned into three disjoint sets: $\{1\}$, the sum decomposable separable permutations, and the skew decomposable separable permutations. This observation allows one to easily enumerate the class.

PROPOSITION 1.4. *The generating function for the separable permutations is*

$$\frac{1 - x - \sqrt{1 - 6x + x^2}}{2},$$

and thus the number of separable permutations of length n is the nth large Schröder number.

Proof. Let f denote the generating function for the class of separable permutations, $f_{\oplus\text{-dec}}$ the generating function for its sum decomposable elements, and $f_{\ominus\text{-dec}}$ the generating function for its skew decomposable elements. As observed above, we have $f = x + f_{\oplus\text{-dec}} + f_{\ominus\text{-dec}}$. Any sum decomposable permutation may be written uniquely as the direct sum of a sum indecomposable permutation and another permutation, so since the separable class is sum closed, we have $f_{\oplus\text{-dec}} = (f - f_{\oplus\text{-dec}})f$. Solving this shows that $f_{\oplus\text{-dec}} = f^2/(1 + f)$, and then by symmetry $f_{\ominus\text{-dec}} = f^2/(1 + f)$, so $f = x + 2f^2/(1 + f)$. Solving this quadratic yields the desired generating function. □

A similar approach gives the generating function for Av(231).

PROPOSITION 1.5. *The generating function for* Av(231) *is*

$$\frac{1 - 2x - \sqrt{1 - 4x}}{2x},$$

and thus the number of 231*-avoiding permutations of length n is the nth Catalan number.*

Proof. Again let f denote the generating function for Av(231), and let $f_{\oplus\text{-dec}}$ and $f_{\ominus\text{-dec}}$, respectively, count the sum and skew decomposable permutations in this class. Since Av(231) is sum closed, by the same logic as in the proof of Proposition 1.4, $f_{\oplus\text{-dec}} = f^2/(1 + f)$. Now note that $\pi \ominus \sigma \in \text{Av}(231)$ if and only if π is decreasing and $\sigma \in \text{Av}(231)$. Thus, every skew decomposable permutation in Av(231) may be written uniquely as $1 \ominus \sigma$ for $\sigma \in \text{Av}(231)$, so $f_{\ominus\text{-dec}} = xf$. Substituting these values into the equation $f = x + f_{\oplus\text{-dec}} + f_{\ominus\text{-dec}}$ yields that $f = x(1 + f)^2$, and solving this quadratic gives the generating function claimed. □

Note that both of these generating functions are nonrational. While it may happen by accident that a particular superclass $\mathcal{C} \supseteq \mathrm{Av}(231)$ has a rational generating function, we expect 'typical' superclasses to have nonrational generating functions. Our main result establishes the converse: if $\mathcal{C}$ is a subclass of the separable permutations that does not contain any of Av(132), Av(213), Av(231), or Av(312), then $\mathcal{C}$ has a rational generating function.

2. *Partial well-order and atomicity*

Many of our arguments depend on the *partial well-order* (*pwo*) property. In the context of the containment order on permutations, a permutation class has the pwo property if it does not contain an infinite antichain. This property has the following well-known consequence which is important to us because it allows us to consider minimal counterexamples within a pwo class.

Proposition 2.1. *The subclasses of a pwo class $\mathcal{C}$ satisfy the minimum condition, that is, every family of subclasses of a pwo class $\mathcal{C}$ has a minimal subclass under inclusion.*

Proof. If there were a family of subclasses with no minimal element, then we could, inductively, find a strictly descending chain

$$\mathcal{C}^1 \supsetneq \mathcal{C}^2 \supsetneq \dots$$

of subclasses of $\mathcal{C}$. For each $i \geqslant 1$, choose $\beta_i \in \mathcal{C}^i \setminus \mathcal{C}^{i+1}$. The set of minimal elements of $\{\beta_1, \beta_2 \dots\}$ is an antichain and therefore finite. Hence, there exists an integer n such that $\{\beta_1, \beta_2, \dots, \beta_n\}$ contains these minimal elements. In particular, $\beta_m \leqslant \beta_{n+1}$ for some $m \leqslant n$, but $\mathcal{C}^{n+1}$ is a class, and therefore $\beta_m \in \mathcal{C}^{n+1} \subset \mathcal{C}^{m+1}$, which is a contradiction. □

For any class $\mathcal{C}$, we define its *sum completion* $\oplus\mathcal{C}$ as the smallest sum-closed class containing $\mathcal{C}$, and we define its *strong completion* $\mathrm{sc}(C)$ as the smallest sum and skew-closed class containing $\mathcal{C}$. In this notation, [**2**, Theorem 2.5] states the following proposition.

Proposition 2.2. *The sum completion and the strong completion of a pwo class are pwo.*

Because the separable permutations are the strong completion of the set $\{1\}$, the separable class is pwo.

Another key concept that we shall require is atomicity. A permutation class is called *atomic* if it cannot be written as the union of two proper subclasses. The notion of atomicity was first studied (in the more general context of relational structures) by Fraïssé [**10**], who established several alternative characterizations of this property. The only characterization we require features in our next proposition. For a proof of this result in the context of permutations, we refer to Atkinson, Murphy, and Ruškuc [**3**, Theorem 1.2].

Proposition 2.3. *If $\mathcal{C}$ is an atomic class, then there is a chain $\alpha_1 \leqslant \alpha_2 \leqslant \dots$ of permutations in $\mathcal{C}$ such that $\mathcal{C} = \mathrm{Cl}(\{\alpha_1, \alpha_2, \dots\})$.*

We refer to such a chain as a *spine* for the class.

3. *Strongly rational classes*

Our main goal is to prove that if a subclass of the separable permutations does not contain Av(231) or any of its symmetries, then it and each of its subclasses have rational generating functions. In this section, we study this powerful property in its own right, beginning by naming it: the permutation class $\mathcal{C}$ is *strongly rational* if it and each of its subclasses have rational generating functions. While strongly rational classes are naturally defined and appear to be the 'correct' context in which to state and prove the tools of this section, they have received virtually no attention before, and many conjectures remain.

PROPOSITION 3.1. *The union and intersection of two strongly rational classes are strongly rational.*

Proof. The intersection of two strongly rational classes is contained in both of them, and so is strongly rational by definition. Now suppose that $\mathcal{C}$ and $\mathcal{D}$ are strongly rational and that $\mathcal{E} \subseteq \mathcal{C} \cup \mathcal{D}$. Since $\mathcal{E} = (\mathcal{C} \cap \mathcal{E}) \cup (\mathcal{D} \cap \mathcal{E})$, we can enumerate it by inclusion–exclusion: the generating function for $\mathcal{E}$ is the generating function for $\mathcal{C} \cap \mathcal{E}$ plus the generating function for $\mathcal{D} \cap \mathcal{E}$ minus the generating function for $\mathcal{C} \cap \mathcal{D} \cap \mathcal{E}$. As $\mathcal{C}$ and $\mathcal{D}$ are strongly rational, each of these three generating functions are rational, so $\mathcal{E}$ has a rational generating function, verifying that $\mathcal{C} \cup \mathcal{D}$ is strongly rational. □

We note that Proposition 3.1 does not hold for classes with rational generating functions in general. Neither does our next proposition, which follows from an argument of Atkinson and Stitt [**4**] first formalized by Murphy [**13**, Chapter 9] (although not in this context).

PROPOSITION 3.2. *Strongly rational classes are partially well ordered.*

Proof. Suppose that the class $\mathcal{C}$ is not pwo. Therefore, it contains an infinite antichain, and in particular contains an infinite antichain $A \subseteq \mathcal{C}$ with at most one member of each length. If $A^{(1)} \neq A^{(2)}$ are two subsets of A, then the two subclasses $\mathcal{C} \cap \mathrm{Av}(A^{(1)})$ and $\mathcal{C} \cap \mathrm{Av}(A^{(2)})$ have different enumerations; in particular, if $A^{(1)}$ and $A^{(2)}$ agree up to permutations of length $n-1$ but disagree on permutations of length n, then $\mathcal{C} \cap \mathrm{Av}(A^{(1)})$ and $\mathcal{C} \cap \mathrm{Av}(A^{(2)})$ will have the same enumeration up to length $n-1$ but will differ on length n permutations. Because A is infinite, it follows that $\mathcal{C}$ has uncountably many subclasses with different generating functions. These generating functions cannot all be rational, so $\mathcal{C}$ is not strongly rational. □

Our next step on the path to more powerful tools is the following proposition.

PROPOSITION 3.3. *If the class $\mathcal{C}$ is strongly rational, then each of the following sets have rational generating functions:*

(1) *the sum indecomposable permutations in $\mathcal{C}$;*
(2) *the sum decomposable permutations in $\mathcal{C}$;*
(3) *the skew indecomposable permutations in $\mathcal{C}$ and*
(4) *the skew decomposable permutations in $\mathcal{C}$.*

Proof. It suffices to prove the claim for the sum indecomposable permutations in $\mathcal{C}$ as the remaining cases follow by symmetry or subtraction. If the claim were false, then, because strongly rational classes are pwo by Proposition 3.2, the minimum condition of Proposition 2.1

shows that any counterexample would have a minimal subclass that was also a counterexample. Choose $\mathcal{C}$ to be such a minimal counterexample. By Proposition 2.2, the class $\oplus\mathcal{C}$ is pwo and so the antichain of minimal elements of the difference $(\oplus\mathcal{C}) \setminus \mathcal{C}$ is a finite set, say $\{\beta_1, \ldots, \beta_m\}$. (These β_i are nothing other than the sum decomposable basis elements of $\mathcal{C}$.) Now decompose each β_i as

$$\beta_i = \beta_{i,1} \oplus \beta_{i,2} \oplus \ldots \oplus \beta_{i,n_i},$$

where each $\beta_{i,j}$ is sum indecomposable. For any permutation π, we define $\mathbf{b}(\pi) = (b_1, \ldots, b_m)$ where each b_i is chosen so that π contains $\beta_{i,1} \oplus \ldots \oplus \beta_{i,b_i}$ but avoids $\beta_{i,1} \oplus \ldots \oplus \beta_{i,b_i} \oplus \beta_{i,b_{i+1}}$. Note that $\mathbf{b}(\pi) \leqslant \mathbf{n} - \mathbf{1} = (n_1 - 1, \ldots, n_m - 1)$ for all permutations $\pi \in \mathcal{C}$. (Here and in what follows we partially order vectors by the *dominance order*, meaning that $(q_1, \ldots, q_m) \leqslant (p_1, \ldots, p_m)$ if and only if $q_i \leqslant p_i$ for all $1 \leqslant i \leqslant m$.)

We need to define a variety of generating functions:

(1) f denotes the generating function for the class $\mathcal{C}$;
(2) for a vector $\mathbf{p}$ of natural numbers, $f_{\mathbf{p}}$ denotes the generating function for all permutations in $\mathcal{C}$ which avoid $\beta_{i,p_i+1} \oplus \ldots \oplus \beta_{i,n_i}$ for all i;
(3) $f_{\oplus\text{-dec}}$ denotes the generating function for the sum decomposable permutations in $\mathcal{C}$;
(4) $f_{\oplus\text{-ind}}$ denotes the generating function for the sum indecomposable permutations in $\mathcal{C}$ and
(5) for a vector $\mathbf{p}$ of natural numbers, $f^{\mathbf{p}}_{\oplus\text{-ind}}$ denotes the generating function for the sum indecomposable permutations in $\mathcal{C}$ with $\mathbf{b}(\pi) = \mathbf{p}$.

Note that $f_{\oplus\text{-ind}}$, the generating function we wish to prove rational, is the sum of the generating functions $f^{\mathbf{p}}_{\oplus\text{-ind}}$ for all $\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{n} - \mathbf{1}$. We begin by claiming that $f^{\mathbf{p}}_{\oplus\text{-ind}}$ is rational for all $\mathbf{0} \leqslant \mathbf{p} < \mathbf{n} - \mathbf{1}$. We establish this claim by induction on the sum of the entries of $\mathbf{p}$. It is clearly true for the base case $f^{\mathbf{0}}_{\oplus\text{-ind}}$, as this function counts sum indecomposable elements of the proper subclass $\{\pi \in \mathcal{C} : \mathbf{b}(\pi) = \mathbf{0}\}$ of $\mathcal{C}$. For larger $\mathbf{p}$, $f^{\mathbf{p}}_{\oplus\text{-ind}}$ can be expressed as the difference between the generating function for sum indecomposable elements of the proper subclass $\{\pi \in \mathcal{C} : \mathbf{b}(\pi) \leqslant \mathbf{p}\}$ of $\mathcal{C}$ (which is rational by our choice of $\mathcal{C}$) and the sum of the generating functions $f^{\mathbf{q}}_{\oplus\text{-ind}}$ for all $\mathbf{0} \leqslant \mathbf{q} < \mathbf{p}$ (which are rational by induction). With this claim established, our goal is only to show that $f^{\mathbf{n}-\mathbf{1}}_{\oplus\text{-ind}}$ is rational.

We now enumerate $\mathcal{C}$, thereby expressing f, which is known to be rational, in terms of the $f^{\mathbf{p}}_{\oplus\text{-ind}}$ and $f_{\mathbf{p}}$ functions. In the resulting equation, $f^{\mathbf{n}-\mathbf{1}}_{\oplus\text{-ind}}$ will be the only term not already known to be rational, yielding a contradiction and completing the proof.

Consider the sum decomposable permutations of $\mathcal{C}$. Each of these can be expressed uniquely as $\sigma \oplus \tau$, where σ is sum indecomposable and so, counting how many of the summands in β_i are contained in σ, we see that $f_{\oplus\text{-dec}}$ is the sum of $f^{\mathbf{p}}_{\oplus\text{-ind}} f_{\mathbf{p}}$ for all vectors $\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{n} - \mathbf{1}$. As $f_{\oplus\text{-ind}}$ is the sum of the generating functions $f^{\mathbf{p}}_{\oplus\text{-ind}}$, we obtain the equation

$$f = f_{\oplus\text{-ind}} + f_{\oplus\text{-dec}} = \sum_{\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{n}-\mathbf{1}} f^{\mathbf{p}}_{\oplus\text{-ind}} + \sum_{\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{n}-\mathbf{1}} f^{\mathbf{p}}_{\oplus\text{-ind}} f_{\mathbf{p}} = \sum_{\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{n}-\mathbf{1}} f^{\mathbf{p}}_{\oplus\text{-ind}} (f_{\mathbf{p}} + 1).$$

Solving for $f^{\mathbf{n}-\mathbf{1}}_{\oplus\text{-ind}}$ shows that

$$f^{\mathbf{n}-\mathbf{1}}_{\oplus\text{-ind}} = \frac{f - \sum_{\mathbf{0} \leqslant \mathbf{p} < \mathbf{n}-\mathbf{1}} f^{\mathbf{p}}_{\oplus\text{-ind}} (f_{\mathbf{p}} + 1)}{f_{\mathbf{n}-\mathbf{1}} + 1}.$$

We have shown that $f^{\mathbf{p}}_{\oplus\text{-ind}}$ is rational for all $\mathbf{0} \leqslant \mathbf{p} < \mathbf{n} - \mathbf{1}$, and the $f_{\mathbf{p}}$ generating functions enumerate subclasses of $\mathcal{C}$, so they are rational by the strong rationality of $\mathcal{C}$. Therefore, every term on the right-hand side is rational, so $f^{\mathbf{n}-\mathbf{1}}_{\oplus\text{-ind}}$ and thus $f_{\oplus\text{-ind}}$ must be rational as well. This contradiction to our choice of $\mathcal{C}$ completes the proof. □

Proposition 3.3 is a stepping stone to a general enumerative result we will need in the proof of our main result. A permutation is said to be *skew-merged* if it is the union of an

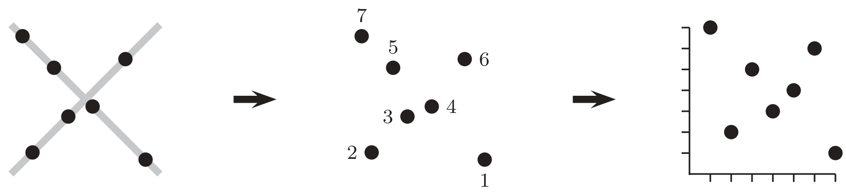

FIGURE 2. *The permutation* 7253461 *can be drawn on an* X.

increasing subsequence and a decreasing subsequence. The class of skew-merged permutations was first studied by Stankova [**14**] in one of the earliest papers on permutation patterns, and later enumerated by Atkinson [**1**]. Stankova proved that the skew-merged permutations have the basis $\{2143, 3412\}$, a result which can also be seen to follow from Földes and Hammer's characterization of split graphs [**9**]. Our interest lies with the class of separable skew-merged permutations:

$$\mathcal{X} = \mathrm{Av}(2143, 2413, 3142, 3412).$$

We label this class $\mathcal{X}$ because, in his thesis, Waton [**15**] showed that these are precisely the permutations that can be 'drawn on an X' via the following procedure: choose, from an X made of right angles which form 45° angles with the axes in the plane, n points, no two lying on a common vertical or horizontal line, and label these points $1, \ldots, n$ reading bottom to top, then record these values reading left to right, as depicted in Figure 2. We may also define the class $\mathcal{X}$ in a manner similar to Propositions 1.2 and 1.3.

PROPOSITION 3.4. *The class $\mathcal{X}$ is the smallest nonempty class $\mathcal{C}$ that contains $\mathcal{C} \oplus 1$, $1 \oplus \mathcal{C}$, $\mathcal{C} \ominus 1$, and $1 \ominus \mathcal{C}$.*

Waton enumerated the class $\mathcal{X}$, obtaining the generating function $(1-3x)/(1-4x+2x^2)$. Later, Elizalde [**8**] constructed a bijection between the class $\mathcal{X}$ and the set of 'almost-increasing permutations' considered by Knuth [**12**, Section 5.4.8, Exercise 8].

For the proof of our main result, we are interested not in the class $\mathcal{X}$ but rather in the $\mathcal{X}$-inflation of a strongly rational class $\mathcal{U}$. This inflation, denoted by $\mathcal{X}[\mathcal{U}]$, can be visualized by taking any permutation in $\mathcal{X}$, drawing it on the X as above, and then replacing each point in this drawing with a set of points corresponding to a permutation in $\mathcal{U}$ in such a way that the relationships between elements belonging to different points (of the permutation from $\mathcal{X}$) are the same as those between the original points. Thus, each point on the original drawing is 'inflated' into a permutation from $\mathcal{U}$. As we show below, such inflations are strongly rational.

THEOREM 3.5. *If $\mathcal{U}$ is a strongly rational class, then $\mathcal{X}[\mathcal{U}]$ is also strongly rational.*

Proof. Let $\mathcal{U}$ be a strongly rational class. It is instructive to first consider the enumeration of $\mathcal{X}[\mathcal{U}]$ itself. Given a sum decomposable permutation in $\mathcal{X}[\mathcal{U}]$, it may decompose in one of two ways, either as a member of $\mathcal{U}_{\oplus\text{-ind}} \oplus \mathcal{X}[\mathcal{U}]$, or as a member of $\mathcal{X}[\mathcal{U}] \oplus \mathcal{U}_{\oplus\text{-ind}}$, or both, where $\mathcal{U}_{\oplus\text{-ind}}$ denotes the set of sum indecomposable elements of $\mathcal{U}$. The intersection of these two sets is $\mathcal{U}_{\oplus\text{-ind}} \oplus \{\mathcal{X}[\mathcal{U}] \cup \epsilon\} \oplus \mathcal{U}_{\oplus\text{-ind}}$, where ϵ denotes the empty permutation. Doing the same for skew decomposable elements of $\mathcal{X}[\mathcal{U}]$ leads us to the equation

$$g = x + 2f_{\oplus\text{-ind}}g - f^2_{\oplus\text{-ind}}(g+1) + 2f_{\ominus\text{-ind}}g - f^2_{\ominus\text{-ind}}(g+1), \qquad (\dagger)$$

where g denotes the generating function for the nonempty permutations in $\mathcal{X}[\mathcal{U}]$, $f_{\oplus\text{-ind}}$ the generating function for $\mathcal{U}_{\oplus\text{-ind}}$, and $f_{\ominus\text{-ind}}$ the generating function for $\mathcal{U}_{\ominus\text{-ind}}$. Solving for g shows that it is indeed rational in $f_{\oplus\text{-ind}}$ and $f_{\ominus\text{-ind}}$, which are themselves elements of $\mathbb{Q}(x)$ by Proposition 3.3. Specifically,

$$g = \frac{x - f_{\oplus\text{-ind}}^2 - f_{\ominus\text{-ind}}^2}{1 - 2f_{\oplus\text{-ind}} + f_{\oplus\text{-ind}}^2 - 2f_{\ominus\text{-ind}} + f_{\ominus\text{-ind}}^2}.$$

Reassuringly, substituting $f_{\oplus\text{-ind}} = f_{\ominus\text{-ind}} = x$ gives us the generating function $(x - 2x^2)/(1 - 4x + 2x^2)$, which, upon adding 1 to count the empty permutation, agrees with Waton's enumeration of $\mathcal{X} = \mathcal{X}[1]$.

To complete the proof, that *all* subclasses of $\mathcal{X}[\mathcal{U}]$ have rational generating functions, we adapt some notation of Brignall, Huczynska, and Vatter [7]. A *property* is any set of permutations, and we say that π *satisfies* the property P if $\pi \in P$. Given a set of properties $\mathcal{P}$, we say that $\mathcal{P}$ is *separable query-complete* if, for all nonempty permutations σ and τ (not necessarily lying in any class) and $P \in \mathcal{P}$, it can be decided whether $\sigma \oplus \tau$ and $\sigma \ominus \tau$ satisfy P given only the knowledge about what properties in $\mathcal{P}$ are satisfied by σ and τ. For example, letting $\oplus$-dec denote the set of sum decomposable permutations, we see that $\{\oplus\text{-dec}\}$ is trivially separable query-complete: assuming that σ and τ are nonempty, $\sigma \oplus \tau$ always satisfies $\oplus$-dec, while $\sigma \ominus \tau$ never satisfies $\oplus$-dec. Similarly, letting $\ominus$-dec denote the set of skew decomposable permutations, $\{\ominus\text{-dec}\}$ is separable query-complete. Also note that, for any permutation β, the set $\{\text{Av}(\delta) : \delta \leqslant \beta\}$ is separable query-complete: $\sigma \oplus \tau$ lies in $\text{Av}(\delta)$ if and only if $\sigma \in \text{Av}(\gamma)$ or $\tau \in \text{Av}(\iota)$ for all $\gamma, \iota \leqslant \delta \leqslant \beta$ satisfying $\gamma \oplus \iota = \delta$.

Returning to the situation at hand, consider an arbitrary subclass $\mathcal{D} \subseteq \mathcal{X}[\mathcal{U}]$. As $\mathcal{U}$ is strongly rational it is pwo by Proposition 3.2. Thus, $\mathcal{X}[\mathcal{U}]$ is contained in the strong completion of a pwo class, and so is itself pwo by Proposition 2.2. Hence, $\mathcal{X}[\mathcal{U}] \setminus \mathcal{D}$ has only a finite number of minimal elements. This set, say B, of minimal elements (which is not a true basis, but rather a 'relative basis') completely defines $\mathcal{D}$ as a subclass of $\mathcal{X}[\mathcal{U}]$ because $\mathcal{X}[\mathcal{U}] \setminus \mathcal{D}$ is the set of permutations of $\mathcal{X}[\mathcal{U}]$ that contain one or more permutations of B. It follows that $\{\text{Av}(\delta) : \delta \in \text{Cl}(B)\}$ is a *finite* separable query-complete set, since it is the union of a finite number of separable query-complete sets. Slightly more generally,

$$\mathcal{P} = \{\oplus\text{-dec}, \ominus\text{-dec}\} \cup \{\text{Av}(\delta) : \delta \in \text{Cl}(B)\}$$

is also a finite separable query-complete set of properties.

For any permutation π, let $\mathcal{P}(\pi)$ denote the set of properties in $\mathcal{P}$ satisfied by π. We introduce three families of generating functions which are defined for any subset $\mathcal{Q} \subseteq \mathcal{P}$:

(1) $f_{\mathcal{Q}}$, the generating function for the set $\{\pi \in \mathcal{U} : \mathcal{P}(\pi) = \mathcal{Q}\}$;
(2) $g_{\mathcal{Q}}$, the generating function for the set $\{\pi \in \mathcal{X}[\mathcal{U}] : \mathcal{P}(\pi) \supseteq \mathcal{Q}\}$;
(3) $h_{\mathcal{Q}}$, the generating function for the set $\{\pi \in \mathcal{X}[\mathcal{U}] : \mathcal{P}(\pi) = \mathcal{Q}\}$.

Our goal, with this notation, is to show that all functions of the form $g_{\mathcal{Q}}$ are rational, since it will then follow that the generating function for $\mathcal{D}$, namely $g_{\{\text{Av}(\beta):\beta\in B\}}$, is rational.

First we claim that the f generating functions are rational. Suppose that $\mathcal{Q}$ contains $\oplus$-dec. Then if $\mathcal{Q}$ also contains $\ominus$-dec, we have that $f_{\mathcal{Q}}$ is either x (counting the trivial permutation) or 0, and trivially rational, so we may suppose that $\mathcal{Q}$ does not contain $\ominus$-dec. The remaining properties in $\mathcal{Q}$ specify exactly which elements of $\text{Cl}(B)$ the permutations must contain and avoid. Therefore, $f_{\mathcal{Q}}$ can be expressed using inclusion–exclusion as a linear combination of generating functions which count sum decomposable permutations in subclasses of $\mathcal{U}$. It then follows from Proposition 3.3 that each $f_{\mathcal{Q}}$ is rational.

Also note that $g_{\mathcal{Q}}$ is the sum of all $h_{\mathcal{R}}$ with $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$. Thus, we can establish that the g generating functions are rational, which will prove the theorem, by showing that the h generating functions are rational.

We are now ready to describe the analogs of the terms of (†) which relate the h generating functions to the f generating functions and to each other. Consider any subset $\mathcal{Q} \subseteq \mathcal{P}$ of properties containing $\oplus$-dec. The permutations satisfying $\mathcal{Q}$ must be sum decomposable, and thus can be expressed as $\sigma \oplus \tau$ where at least one of σ or τ is a sum indecomposable permutation in $\mathcal{U}$. Following our derivation of (†), such an $h_{\mathcal{Q}}$ can be expressed as a linear combination of terms of three forms:

(1) $f_{\mathcal{R}} h_{\mathcal{S}}$ with $\oplus$-dec $\notin \mathcal{R}$, which count permutations from $\mathcal{U}_{\oplus\text{-ind}} \oplus \mathcal{X}[\mathcal{U}]$;
(2) $h_{\mathcal{S}} f_{\mathcal{T}}$ with $\oplus$-dec $\notin \mathcal{T}$, which count permutations from $\mathcal{X}[\mathcal{U}] \oplus \mathcal{U}_{\oplus\text{-ind}}$ and
(3) $f_{\mathcal{R}} h_{\mathcal{S}} f_{\mathcal{T}}$ with $\oplus$-dec $\notin \mathcal{R}, \mathcal{T}$, which count permutations from $\mathcal{U}_{\oplus\text{-ind}} \oplus \mathcal{X}[\mathcal{U}] \oplus \mathcal{U}_{\oplus\text{-ind}}$;

the latter occurring with negative coefficients to correct for over-counting. Similarly, if $\mathcal{Q} \subseteq \mathcal{P}$ contains $\ominus$-dec, then $h_{\mathcal{Q}}$ can be expressed as a linear combination of terms of the form $f_{\mathcal{R}} h_{\mathcal{S}}$ with $\ominus$-dec $\notin \mathcal{R}$, $h_{\mathcal{S}} f_{\mathcal{T}}$ with $\ominus$-dec $\notin \mathcal{T}$, and $f_{\mathcal{R}} h_{\mathcal{S}} f_{\mathcal{T}}$ with $\ominus$-dec $\notin \mathcal{R}, \mathcal{T}$. As no permutation can be both sum decomposable and sum indecomposable, there is only one more case, where neither $\oplus$-dec nor $\ominus$-dec lie in $\mathcal{Q}$. However, in this case the only permutations in $\mathcal{X}[\mathcal{U}]$ that can satisfy precisely the properties $\mathcal{Q}$ are those of $\mathcal{U}$, so here $h_{\mathcal{Q}} = f_{\mathcal{Q}}$.

Therefore, letting $\mathbf{h}$ denote the column vector consisting of the $h_{\mathcal{Q}}$ generating functions, there is some matrix M of rational functions in $\mathbb{Q}(x)$ and some constant vector $\mathbf{v}$ over $\mathbb{Q}(x)$ such that $\mathbf{h} = M\mathbf{h} + \mathbf{v}$. Since all of our generating functions enumerate nonempty permutations, the entries of M all have zero constant term. Hence, $I - M$ is invertible over $\mathbb{Q}(x)$, and thus each entry of $\mathbf{h}$ is a rational function, proving the theorem. □

We now have the following immediate corollary.

Corollary 3.6. *If $\mathcal{C}$ is a strongly rational class, then $\oplus\mathcal{C}$ and $\ominus\mathcal{C}$ are strongly rational, and if $\mathcal{C}$ and $\mathcal{D}$ are strongly rational, then $\mathcal{C} \oplus \mathcal{D}$ and $\mathcal{C} \ominus \mathcal{D}$ are strongly rational.*

Proof. The statements about skew sums follow by symmetry, so we prove only the statements for direct sums. Since $\oplus\mathcal{C} \subseteq \mathcal{X}[\mathcal{C}]$, the first part of the corollary follows from Theorem 3.5. For the second part of the corollary, Proposition 3.1 shows that $\mathcal{C} \cup \mathcal{D}$ is strongly rational, so $\oplus(\mathcal{C} \cup \mathcal{D})$ is strongly rational by the first part of the corollary, and the result follows from observing that $\mathcal{C} \oplus \mathcal{D} \subseteq \oplus(\mathcal{C} \cup \mathcal{D})$. □

4. *Proof of the main result*

With the machinery of Sections 2 and 3 developed, we are now ready to state and prove our main result.

Theorem 4.1. *If $\mathcal{C}$ is a subclass of the separable permutations that does not contain any of* Av(132), Av(213), Av(231), *or* Av(312), *then $\mathcal{C}$ has a rational generating function.*

Proof. Suppose otherwise. Because the separable class is partially well ordered, its subclasses satisfy the minimum condition of Proposition 2.1, and we can therefore choose among all the counterexamples a minimal class $\mathcal{C}$. We use two properties of $\mathcal{C}$ repeatedly:

(i) all proper subclasses of $\mathcal{C}$ have rational generating functions because $\mathcal{C}$ is a minimal counterexample, and thus
(ii) $\mathcal{C}$ is atomic because otherwise it would be the union of two strongly rational classes and hence strongly rational by Proposition 3.1.

Our proof that $\mathcal{C}$ does not exist begins by ruling out some easy cases. The easiest possibility to rule out is when $\mathcal{C}$ is either a sum or skew of two proper subclasses, which is eliminated by Corollary 3.6.

Next we dispense with the case where $\mathcal{C}$ is sum closed (the case where $\mathcal{C}$ is skew closed is similar). In this case, we define $\mathcal{C}_{\oplus\text{-ind}}$ as the set of sum indecomposable elements of $\mathcal{C}$. It must be the case that $\text{Cl}(\mathcal{C}_{\oplus\text{-ind}}) = \mathcal{C}$, for otherwise it would be a proper subclass of $\mathcal{C}$ and so strongly rational; but then, again by Corollary 3.6, the sum closure of $\text{Cl}(\mathcal{C}_{\oplus\text{-ind}})$ would also be strongly rational and so its subclass $\mathcal{C}$ would be strongly rational, which is a contradiction. In the same way, $\text{Cl}(\mathcal{C}_{\ominus\text{-ind}}) = \mathcal{C}$. In other words, every permutation in $\mathcal{C}$ is contained in both a sum indecomposable permutation and a skew indecomposable permutation of $\mathcal{C}$.

Consider any spine $\alpha_1, \alpha_2, \ldots$ of $\mathcal{C}$ (in the sense of Proposition 2.3). Clearly, $1 \oplus \alpha_1, 1 \oplus \alpha_2, \ldots$ is also a spine for $\mathcal{C}$ because we are assuming that $\mathcal{C}$ is sum closed, and since $\text{Cl}(\mathcal{C}_{\oplus\text{-ind}}) = \mathcal{C}$, each of these permutations is contained in a sum indecomposable element of $\mathcal{C}$. However, because $\mathcal{C}$ contains only separable permutations, the sum indecomposable permutations in $\mathcal{C}$ (of length at least 2) are precisely the skew decomposable permutations. The only way that $1 \oplus \alpha_i$ can be contained in a sum indecomposable element of $\mathcal{C}$ is if it embeds completely into a single skew component† of such a permutation. Thus, for all i, either $(1 \oplus \alpha_i) \ominus 1$ or $1 \ominus (1 \oplus \alpha_i)$ lies in $\mathcal{C}$. As one or the other of these possibilities must occur infinitely often, we see that either $\mathcal{C} = 1 \ominus \mathcal{C}$ or $\mathcal{C} = \mathcal{C} \ominus 1$. Proposition 1.3 now shows that $\mathcal{C}$ contains Av(231) or a symmetry, Av(312), which contradicts our choice of $\mathcal{C}$. Similarly, we reach a contradiction if $\mathcal{C}$ is skew closed.

For the remainder of the proof, we may therefore take $\mathcal{C}$ to be neither a sum or skew of two proper subclasses nor to be sum or skew closed. To complete the proof, we shall find a proper subclass $\mathcal{U} \subsetneq \mathcal{C}$ for which $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$. This will indeed be a contradiction because, by the minimality of $\mathcal{C}$, $\mathcal{U}$ will be strongly rational and therefore, by Theorem 3.5, $\mathcal{X}[\mathcal{U}]$ will also be strongly rational.

We now construct a finite collection of proper subclasses of $\mathcal{C}$ whose union will yield the desired $\mathcal{U}$. To do this, we shall rely on the following characterization of subclasses of $\mathcal{X}[\mathcal{U}]$, which follows trivially from Proposition 3.4.

PROPOSITION 4.2. *Given classes $\mathcal{C}$ and $\mathcal{U}$, we have $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$ if and only if, for every $\pi \in \mathcal{C}$, one of the following holds (for nonempty γ and τ) :*

(1) $\pi \in \mathcal{U}$;
(2) $\pi = \gamma \oplus \tau$ *with* $\gamma \in \mathcal{U}$ *or* $\tau \in \mathcal{U}$ *or*
(3) $\pi = \gamma \ominus \tau$ *with* $\gamma \in \mathcal{U}$ *or* $\tau \in \mathcal{U}$.

With the aim of mimicking the structural decomposition provided by this proposition, we begin by defining

$$\mathcal{C}_{SW} = \{\sigma \in \mathcal{C} : \sigma \oplus \mathcal{C} \subseteq \mathcal{C}\}.$$

Note that $\mathcal{C}_{SW}$ is a proper subclass of $\mathcal{C}$ because $\mathcal{C}$ is not sum closed. In fact, $\mathcal{C}_{SW}$ is the maximum subclass $\mathcal{F}$ of $\mathcal{C}$ such that $\mathcal{F} \oplus \mathcal{C} \subseteq \mathcal{C}$. We similarly define $\mathcal{C}_{NW}$ maximal such that $\mathcal{C}_{NW} \ominus \mathcal{C} \subseteq \mathcal{C}$, $\mathcal{C}_{NE}$ maximal such that $\mathcal{C} \oplus \mathcal{C}_{NE} \subseteq \mathcal{C}$, and $\mathcal{C}_{SE}$ maximal such that $\mathcal{C} \ominus \mathcal{C}_{SE} \subseteq \mathcal{C}$. As $\mathcal{C}$ is neither sum nor skew closed, these are all proper subclasses of $\mathcal{C}$ (and may indeed be empty). These are the first four classes that will be placed within $\mathcal{U}$.

Consider any sum decomposable element π of $\mathcal{C}$ and write $\pi = \gamma \oplus \tau$ in an arbitrary fashion. If $\gamma \in \mathcal{C}_{SW}$ or $\tau \in \mathcal{C}_{NE}$, then the conditions of Proposition 4.2 are already met. So suppose

†If $\pi = \sigma_1 \ominus \ldots \ominus \sigma_m$, where each σ_i is skew indecomposable, $\sigma_1, \ldots, \sigma_m$ are the *skew components* of π.

now that we have $\gamma \notin \mathcal{C}_{SW}$ and $\tau \notin \mathcal{C}_{NE}$ with $\gamma \oplus \tau \in \mathcal{C}$. Define

$$\mathcal{E}_\gamma = \{\sigma \in \mathcal{C} : \gamma \oplus \sigma \in \mathcal{C}\}.$$

Clearly, $\mathcal{E}_\gamma$ is a subclass of $\mathcal{C}$, and it is proper because $\gamma \notin \mathcal{C}_{SW}$, so $\gamma \oplus \mathcal{C} \nsubseteq \mathcal{C}$. Also note that $\tau \in \mathcal{E}_\gamma$. Now define

$$\mathcal{D}_\gamma = \{\sigma \in \mathcal{C} : \sigma \oplus \mathcal{E}_\gamma \subseteq \mathcal{C}\}.$$

Again, $\mathcal{D}_\gamma$ is a subclass of $\mathcal{C}$ and it is proper since $\tau \in \mathcal{E}_\gamma$ and $\tau \notin \mathcal{C}_{NE}$, so $\mathcal{C} \oplus \mathcal{E}_\gamma \nsubseteq \mathcal{C}$. From our definitions, it follows that $\mathcal{D}_\gamma \oplus \mathcal{E}_\tau \subseteq \mathcal{C}$, and this containment is proper because $\mathcal{C}$ is not the sum of two proper subclasses. Lastly, but importantly, note that $\pi = \gamma \oplus \tau \in \mathcal{D}_\gamma \oplus \mathcal{E}_\gamma$.

While there may be infinitely many permutations γ of this type, the number of *distinct* classes $\mathcal{D}_\gamma$ is finite. To see this, let B denote the (finite) basis of $\mathcal{C}$ and consider the sets $\mathrm{Cl}(\gamma) \cap \mathrm{Cl}(B)$, of which there are but a finite number. Suppose that we have two of them that happen to be equal, say $\mathrm{Cl}(\gamma) \cap \mathrm{Cl}(B) = \mathrm{Cl}(\overline{\gamma}) \cap \mathrm{Cl}(B)$. Now a permutation σ fails to lie in $\mathcal{E}_\gamma$ if and only if $\gamma \oplus \sigma \notin \mathcal{C}$. But this happens if and only if $\gamma \oplus \sigma$ contains some $\beta_1 \oplus \beta_2 \in B$ with $\beta_1 \leqslant \gamma$ and $\beta_2 \leqslant \sigma$. This means that $\beta_1 \in \mathrm{Cl}(\gamma) \cap \mathrm{Cl}(B) = \mathrm{Cl}(\overline{\gamma}) \cap \mathrm{Cl}(B)$ and so $\beta_1 \leqslant \overline{\gamma}$. In turn this implies that $\overline{\gamma} \oplus \sigma \notin \mathcal{C}$ and hence σ fails to lie in $\mathcal{E}_{\overline{\gamma}}$ also. In other words $\mathcal{E}_\gamma = \mathcal{E}_{\overline{\gamma}}$. But then $\mathcal{D}_\gamma = \mathcal{D}_{\overline{\gamma}}$ also.

Therefore, in addition to $\mathcal{C}_{SW}$, $\mathcal{C}_{SE}$, $\mathcal{C}_{NW}$, and $\mathcal{C}_{NE}$, we include in $\mathcal{U}$ the finitely many classes $\mathcal{D}_\gamma \oplus \mathcal{E}_\gamma$ arising from decompositions of this type. By repeating an analogous argument for skew decompositions $\gamma \ominus \tau$ with $\gamma \notin \mathcal{C}_{NW}$ and $\tau \notin \mathcal{C}_{SE}$, we again find finitely many classes, and also include these in $\mathcal{U}$. Since $\mathcal{C}$ is atomic, $\mathcal{C}$ cannot be equal to a finite union of proper subclasses, so $\mathcal{U} \neq \mathcal{C}$. This choice of $\mathcal{U}$ ensures that the conditions of Proposition 4.2 are met, and thus that $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$, establishing the desired contradiction, and proving the theorem. □

5. *Open problems*

The most obvious question is the converse to our main result: is there a subclass of the separable permutations containing Av(231) which has a rational generating function? In fact, we are not aware of any finitely based permutation class, separable or otherwise, which contains Av(231) and has a rational generating function.

More generally, we are hopeful that the notion of strongly rational classes introduced herein will prove relevant to future investigations of permutation classes. One question, inspired by Theorem 3.5, would be: is there a natural characterization of the classes $\mathcal{C}$ such that $\mathcal{C}[\mathcal{U}]$ is strongly rational for all strongly rational classes $\mathcal{U}$? A positive answer to this question would lend hope to the possibility of a characterization of the strongly rational classes themselves.

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