Simple permutations and pattern restricted permutations

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Abstract

A simple permutation is one that does not map any non-trivial interval onto an interval. It is shown that, if the number of simple permutations in a pattern restricted class of permutations is finite, the class has an algebraic generating function and is defined by a finite set of restrictions. Some partial results on classes with an infinite number of simple permutations are given. Examples of results obtainable by the same techniques are given; in particular it is shown that every pattern restricted class properly contained in the 132-avoiding permutations has a rational generating function.

1 Introduction and definitions

In [14] Simion and Schmidt managed to enumerate the number of permutations of each length that avoided some arbitrary given set of permutation patterns of length 3. Their paper began the systematic study by many authors [2, 5, 6, 7, 8, 10, 11, 15] of sets of permutations characterised by a set

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of avoidance conditions. The techniques in these papers tend to be tailored to the particular avoidance conditions at hand and very little in terms of a general theory has yet emerged. In this paper we shall go some way towards developing a general strategy for carrying out enumeration, and for answering other structural questions about restricted permutations.

The principal tool in our work is the notion of a simple permutation (defined below). We shall show that a knowledge of the simple permutations in a pattern restricted class is often the key to understanding enough of its structure to carry out an enumeration and to answer the related question of whether a finite number of restrictions suffices to define the class. Our results completely answer both these questions when the number of simple permutations in the class is finite but they also have implications in more general cases.

Our paper is laid out as follows. The remainder of this section defines the necessary terminology including the definition of a simple permutation. Then, in section 2, we explain how arbitrary permutations are built from simple ones and how this impacts on the minimal restrictions of a pattern closed class. Section 3 gives a key property of simple permutations that we exploit in the following section when discussing the number of restrictions. The core section is section 5. There we show that the hypothesis of a finite number of simple permutations enables one to solve the enumeration problem (in theory and in practice). Section 6 gives some examples of how our techniques can be applied and we conclude with an overview and some unsolved problems.

A permutation π is a bijective function from $[n] = \{1, 2, \ldots, n\}$ to $[n]$ for some natural number n which is called the degree, or sometimes length, of π. To specify a permutation explicitly we usually write down the sequence of its values. Sets of permutations are denoted by calligraphic letters, $\mathcal{A}$, $\mathcal{B}$ etc. The set of all permutations is denoted $\mathcal{S}$, and $\mathcal{S}_n$ denotes the set of all permutations of length n. If $\mathcal{A}$ is a set of permutations, then A is the ordinary generating function for $\mathcal{A}$, that is:

$$A(x) = \sum_{n=1}^{\infty} |\mathcal{A} \cap \mathcal{S}_n| x^n.$$

The *involvement* (sometimes called pattern containment) relation on $\mathcal{S}$ is a partial order $\preceq$ on $\mathcal{S}$ defined as follows: $\alpha \preceq \beta$ if and only if there is a subsequence of the sequence of values of β whose relative ordering is the same as the sequence of all values of α. Thus $231 \preceq 31524$ because the latter contains the subsequence 352 whose relative ordering is the same as that in 231. The relative ordering of a sequence will sometimes be called its *pattern*. Thus, any finite sequence without repetitions from a linearly ordered set has a unique pattern which is a permutation of the same length.

A *pattern class*, or simply *class*, is a collection of permutations closed downwards under $\preceq$. If $\mathcal{A}$ is a class and $\pi \notin \mathcal{A}$, then no element of $\mathcal{A}$ involves π. In this case we say that π is a *restriction* of $\mathcal{A}$. If in addition π is minimal with respect to $\preceq$ among the restrictions of $\mathcal{A}$, then we say that π is a *basic restriction* of $\mathcal{A}$. The set of basic restrictions of $\mathcal{A}$ is called the *basis* of $\mathcal{A}$ and denoted $\mathbf{basis}(A)$. Thus we have:

$$\mathcal{A} = \bigcap_{\pi \in \mathbf{basis}(\mathcal{A})} \{\theta : \pi \not\preceq \theta\}.$$

If $\mathcal{C}$ is any set of permutations and $\tau_1, \tau_2, \ldots, \tau_k$ are permutations, then we denote the subset of $\mathcal{C}$ consisting of those permutations involving none of $\tau_1, \tau_2, \ldots, \tau_k$ by $\mathcal{C}\langle \tau_1, \tau_2, \ldots, \tau_k \rangle$. With this notation we could also write:

$$\mathcal{A} = \bigcap_{\pi \in \mathbf{basis}(\mathcal{A})} \mathcal{S}\langle \pi \rangle$$

or simply $\mathcal{A} = \mathcal{S}\langle \pi_1, \pi_2, \ldots, \pi_m \rangle$ where the sequence $\pi_1, \pi_2, \ldots, \pi_m$ is a listing of $\mathbf{basis}(\mathcal{A})$.

As an introduction to the central concept of this paper, notice that the permutation 2647513 maps the interval 2..5 onto the interval 4..7. In other words, it has a *segment* (set of consecutive positions) whose values form a *range* (set of consecutive values). Such a segment is called a *block* of the permutation. Every permutation has singleton blocks, together with the block $1..n$. If these are the only blocks the permutation is called *simple*. If $\mathcal{A}$ is a set of permutations, then $\mathbf{Si}(\mathcal{A})$ denotes the set of simple permutations that belong to A.

Simple permutations are the main focus of the paper. The simple permutations of small degree are $1, 12, 21, 2413, 3142$. There are 6 simple permutations of length 5, 46 of length 6 and for large n their number is asymptotic to $n!/e^2$ [1].

Our intent is to show that the simple permutations of a pattern class are a key determinant of its structure. This is particularly true when the class has only finitely many simple permutations. The following summary of results proved later in the paper gives a broad idea of what can be achieved by our approach.

- Every pattern class that contains only finitely many simple permutations has a finite basis and an algebraic generating function.
- Every pattern class that contains only finitely many simple permutations and does not contain the permutation $n(n-1)\cdots 321$ for some n has a rational generating function.
- Every proper subclass of the class with basis $\{132\}$ has a rational generating function.

As we shall see in the arguments leading to the proof of Theorem 8, simple permutations provide the foundations of a framework for dealing with permutation classes in an algebraic way.

2 Block decompositions and the wreath product

Suppose that $\pi \in \mathcal{S}_k$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{S}$. Define the *inflation* of π by $\alpha_1, \alpha_2, \ldots, \alpha_k$ to be the permutation obtained by replacing each element p_i of π by a block whose pattern is α_i (for $1 \leq i \leq k$) so that the relative ordering of the blocks is the same as the relative ordering of the corresponding elements of π. That is, the ordering within a block is determined by the ordering of the corresponding α_i, and the ordering between blocks is determined by π. We denote the resulting permutation by:

$$\pi[\alpha_1, \alpha_2, \ldots, \alpha_k].$$

For example,

$$(213)[21, 312, 4123] = 54\,312\,9678.$$

We also extend this notation to sets defining

$$\pi[\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k]$$

as the set of all permutations of the form $\pi[\alpha_1, \alpha_2, \ldots, \alpha_k]$ with $\alpha_i \in \mathcal{A}_i$.

Inflation is a localized version of the wreath product construction introduced in [4]. Namely, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$, then:

$$\mathcal{A} \wr \mathcal{B} = \{\alpha[\beta_1, \beta_2, \ldots, \beta_k] : \alpha \in \mathcal{A},\ \beta_1, \beta_2, \ldots, \beta_k \in \mathcal{B}\}.$$

For example, if $\mathcal{I}$ is the set of all increasing permutations, and $\mathcal{D}$ the set of all decreasing permutations (i.e. all permutations of the form $n(n-1)\cdots 321$ for some n), then $\mathcal{I} \wr \mathcal{D}$ consists of the *layered* permutations, such as 321465 whose sequence of values are obtained from $12\cdots n$ by dividing it into some number of intervals and reversing each interval.

We say that a set $\mathcal{X}$ of permutations is *wreath-closed* if $\mathcal{X} = \mathcal{X} \wr \mathcal{X}$. The *wreath closure* $\mathbf{wc}(\mathcal{X})$ of a set $\mathcal{X}$ of permutations, is the smallest wreath-closed set of permutations that contains $\mathcal{X}$. The wreath product operation is associative and so, if we define $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_{n+1} = \mathcal{X} \wr \mathcal{X}_n$, then $\mathbf{wc}(\mathcal{X}) = \cup_{n=1}^{\infty} \mathcal{X}_n$.

Proposition 1 *A class is wreath-closed if and only if its basis consists entirely of simple permutations.*

Proof: Let a wreath-closed class $\mathcal{A}$ be given, and suppose that α were a nonsimple basic restriction of $\mathcal{A}$. Thus α has a non-trivial block decomposition, say, $\alpha = \beta[\sigma_1, \sigma_2, \ldots, \sigma_k]$. But as each of β and the σ_i are properly involved in α they all belong to $\mathcal{A}$. Hence $\alpha \in \mathcal{A} \wr \mathcal{A} = \mathcal{A}$ which is impossible.

Conversely, if all the basis elements of $\mathcal{A}$ were simple, then $\mathcal{A}$ could only fail to be wreath-closed if there were permutations $\pi, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{A}$ but with $\pi[\alpha_1, \alpha_2, \ldots, \alpha_k] \notin \mathcal{A}$. The latter permutation would then involve some basis element of $\mathcal{A}$. However, every simple subpermutation of $\pi[\alpha_1, \alpha_2, \ldots, \alpha_k]$ must be involved in one of $\pi, \alpha_1, \alpha_2, \ldots, \alpha_k$ since any involvement including more than one element from a single α_i must occur entirely within α_i otherwise we would obtain a non-trivial block decomposition of a simple permutation. ■

The following proposition establishes that every permutation has a canonical representation as an inflation of a simple permutation. Before stating it we need two definitions. A permutation is said to be *plus-indecomposable* if it cannot be expressed as $(12)[\alpha, \beta]$ and *minus-indecomposable* if it cannot be expressed as $(21)[\alpha, \beta]$.

Proposition 2 *Let $\sigma \in \mathcal{S}$. There is a unique permutation $\pi \in \mathbf{Si}(\mathcal{S})$ and sequence $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{S}$ such that*

$$\sigma = \pi[\alpha_1, \alpha_2, \ldots, \alpha_k].$$

If $\pi \neq 12, 21$, then $\alpha_1, \alpha_2, \ldots, \alpha_k$ are also uniquely determined by σ. If $\pi = 12$ or 21, then α_1, α_2 are unique so long as we require that α_1 is plus-indecomposable or minus-indecomposable respectively.

Proof: We consider the maximal proper blocks of σ. Suppose that two such, say A and B, have nonempty intersection. Since the union of A and B is not a proper block, neither the segments nor the ranges represented by A and B can be interior intervals of $[n]$. So, in this case, $\sigma = (12)[\alpha, \beta]$ or $\sigma = (21)[\alpha, \beta]$ and, provided that we take α to be plus-indecomposable in the former case and minus-indecomposable in the latter, this expression is unique.

In the remaining cases, the maximal proper blocks of σ are disjoint. By maximality, the pattern they define is simple, and the structure of each block is uniquely determined. ∎

The following consequence is readily deduced.

Corollary 3 *Let $\mathcal{A}$ be a wreath-closed class. Then*

$$\mathcal{A} = \mathbf{wc}(\mathbf{Si}(\mathcal{A})).$$

3 Simple subpermutations

This section is devoted to the proof of a result which, although of interest in itself, is given more for its use in the finite basis results appearing later. We shall prove that, in every simple permutation, we can find either one point or two points which, if deleted, yield another simple permutation. In fact, a slightly stronger result is proved and to state it we need the following:

Definition 1 *The following simple permutations are called* exceptional*:*

(i) $2\,4\,6\,\ldots\,(2m)\,1\,3\,5\,\ldots(2m-1)$

(ii) $(2m-1)\,(2m-3)\,\ldots\,1\,(2m)\,(2m-2)\,\ldots\,2$

(iiii) $(m+1)\,1\,(m+2)\,2\,\ldots\,(2m)\,m$

(iv) $m\,(2m)\,(m-1)\,(2m-2)\,\ldots\,1\,(m+1)$

where $m \geq 2$ *in all cases. Using reversal and inversion the last three of these can be obtained from the first.*

Notice that, if we remove the symbols $2m-1$ and $2m$ from the first two of these, we obtain another exceptional (and simple) permutation; and likewise if we remove the symbols in the last two positions from the third and fourth of these.

Theorem 4 *If* π *is simple, then either there is a one point deletion that is also simple or* π *is exceptional (in which case it has a two point deletion that is simple).*

Proof: Associated with every permutation π of $[n]$ is a partially ordered set, or poset, $P(\pi)$ on the set $[n]$ where the order relation is defined by

$$x \ll y \text{ if and only if } x \leq y \text{ and } \pi_x \leq \pi_y.$$

The poset $P(\pi)$ is of dimension 2. Conversely every poset of dimension 2 is of this form and determines a permutation π to within permutational inverse.

In their paper [13] Schmerl and Trotter define a poset to be *indecomposable* if it has no subset I (except for singletons and the entire set) with the property that every two elements $i, j \in I$ are ordered with respect to elements not in I in exactly the same way. If a permutation π is not simple, then any non-trivial block I of π is a subset of $P(\pi)$ which witnesses its non-indecomposability. So, for the posets $P(\pi)$, simplicity of π and indecomposability of $P(\pi)$ are equivalent notions. Furthermore, if π has the property that it is simple but all of its one point deletions are not simple, then $P(\pi)$ is *critically indecomposable* in the sense of [13].

Schmerl and Trotter classified all the critically indecomposable posets. There are two of every even size greater than or equal to 4. Both of them are of dimension 2 and so, with inverses, determine 4 permutations of each even degree. By directly comparing the definitions of the critically indecomposable partially ordered sets found on page 197 of [13] with the exceptional permutations listed above, it will be evident that the permutations which we have labelled exceptional are indeed the only simple permutations that do not have a one point deletion which is also simple. ■

4 Finite basis results

Proposition 5 *Any wreath-closed class that contains only finitely many simple permutations is determined by a finite set of restrictions.*

Proof: Let $\mathcal{A}$ be such a class. By Proposition 1 the basis of $\mathcal{A}$ consists entirely of simple permutations. Suppose that $\sigma \in \mathcal{S}_n$ is such a permutation. By Theorem 4, σ involves a simple permutation π where $\pi \in \mathcal{S}_{n-1}$ or $\pi \in S_{n-2}$. Since σ is a basis element, $\pi \in \mathcal{A}$. Thus the length of σ is at most two more than the length of the longest simple permutation in $\mathcal{A}$. Hence $\mathcal{A}$ is finitely based. ■

In the examples we will show that in some circumstances we can obtain a similar result for some classes with infinitely many simple permutations. However, of greater interest is the fact that we can drop the hypothesis that the class be wreath-closed.

In order to strengthen Proposition 5 we make use of a result of Higman from [9]. For completeness we first state a specialization of Higman's result which is sufficient for our purposes. Recall that a partially ordered set is said to be *well quasi-ordered* if it contains no infinite descending chain, and no infinite antichain.

Let $\mathcal{P}$ be a partially ordered set with ordering $\leq$, and let $f : \mathcal{P}^n \to \mathcal{P}$ be a function. Then, in a slight modification of Higman's terminology, $\leq$ is a *divisibility order* for f, if:

- f is order preserving, and
- for all $p \in P$ and any sequence $x \in \mathcal{P}^n$ in which p occurs, $p \leq f(x)$.

Theorem 6 (Higman) *Let a partially ordered set $\mathcal{P}$ with order relation $\leq$, and finitely many functions $f_i : \mathcal{P}^{n_i} \to \mathcal{P}$ be given. If $\leq$ is a divisibility order for each f_i, then the closure of any finite subset of $\mathcal{P}$ under this collection of functions is well quasi-ordered.*

Corollary 7 *Any wreath-closed class that contains only finitely many simple permutations is well quasi-ordered under involvement.*

Proof: Let $\mathcal{F}$ be a finite set of simple permutations, and let $\mathcal{A} = \mathbf{wc}(\mathcal{F})$. We view $\mathcal{A}$ as an algebra with an operator $\phi : \mathcal{A}^k \to \mathcal{A}$ for each $\phi \in \mathcal{F} \cap \mathcal{S}_k$. Specifically:

$$\phi(\alpha_1, \alpha_2, \ldots, \alpha_k) = \phi[\alpha_1, \alpha_2, \ldots, \alpha_k].$$

So, with these operations, $\mathcal{A}$ is generated by 1. These operations respect the relation $\preceq$ in each argument, and hence preserve $\preceq$. This is easy to see as a block decomposition obtained by replacing one block α with another block α' where $\alpha \preceq \alpha'$ involves the original block decomposition by simply taking all the elements from the other blocks, and those element from the block α' representing a copy of α. Furthermore, by the very definition of inflation, $\alpha_i \preceq \phi[\alpha_1, \alpha_2, \ldots, \alpha_k]$ and so $\preceq$ is a divisibility order for each ϕ. Thus by Higman's theorem $\mathcal{A}$ is well quasi-ordered under involvement. ■

Finally we obtain the promised strengthening of Proposition 5.

Theorem 8 *Any class that contains only finitely many simple permutations is determined by a finite set of restrictions (i.e. is finitely based).*

Proof: Let $\mathcal{C}$ be such a class and let $\mathcal{A}$ be its wreath closure. By Proposition 5, $\mathcal{A}$ is finitely based. A sufficient set of restrictions for $\mathcal{C}$ consists of the basis of $\mathcal{A}$ together with the minimal elements of $\mathcal{A}$ not belonging to $\mathcal{C}$. As $\mathcal{A}$ is well quasi-ordered this latter set is finite, and so $\mathcal{C}$ is determined by a finite set of restrictions. ■

This theorem has been proved independently by Murphy [12]. Our original proof (and the proof of [12]) was rather complicated. We thank Dr. Murphy for pointing out reference [13] which removes most of the complexities.

5 Enumeration results

In this section we develop techniques for studying the generating function of a pattern class if we know its simple permutations. Our main goal is the following result:

Theorem 9 *The generating function of every class that contains only finitely many simple permutations is algebraic.*

Our techniques are constructive in the sense that they can compute (a polynomial satisfied by) the generating function if we are given the simple permutations of the class and its basis. In broad terms our method is to find a structural decomposition first in the case of a wreath-closed class and then in general. From such a decomposition we then read off a set of algebraic equations for the generating function.

Before giving the first structural decomposition we introduce the notation $\mathcal{A}_+, \mathcal{A}_-$ to stand for the set of plus-indecomposable and minus-indecomposable permutations of a class $\mathcal{A}$. Proposition 2 shows that:

Lemma 10 *Suppose that a class $\mathcal{A}$ is wreath-closed, contains the permutations* 12 *and* 21 *(this avoids trivialities), and that* $\mathbf{Si}(\mathcal{A})_{\geq 4} = \mathcal{F}$. *Then*

$$\begin{array}{rcl} \mathcal{A} & = & \{1\} \cup (12)[\mathcal{A}_+, \mathcal{A}] \cup (21)[\mathcal{A}_-, \mathcal{A}] \cup \bigcup_{\pi \in \mathcal{F}} \pi[\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}] \\ \mathcal{A}_+ & = & \{1\} \cup (21)[\mathcal{A}_-, \mathcal{A}] \cup \bigcup_{\pi \in \mathcal{F}} \pi[\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}] \\ \mathcal{A}_- & = & \{1\} \cup (12)[\mathcal{A}_+, \mathcal{A}] \cup \bigcup_{\pi \in \mathcal{F}} \pi[\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}]. \end{array}$$

and all these unions are disjoint.

Passing to generating functions $A = A(x), A_+ = A_+(x), A_- = A_-(x), F = F(x)$, these decompositions become:

$$\begin{array}{rcl} A & = & x + (A_+ + A_-)A + F(A) \\ A_+ & = & x + A_- A + F(A) \\ A_- & = & x + A_+ A + F(A). \end{array} \qquad (1)$$

This system of equations is, in itself, useful for enumerative purposes. However, by eliminating A_+ and A_- we obtain:

Theorem 11 *Let $\mathcal{A}$ be a wreath-closed class, with generating function A, and suppose that the generating function for* $\mathbf{Si}(\mathcal{A})_{\geq 4}$ *is F. Then:*

$$A^2 + (F(A) - 1 + x)A + F(A) + x = 0.$$

Corollary 12 *The generating function of a wreath-closed class $\mathcal{A}$ is algebraic if and only if the generating function of* $\mathbf{Si}(\mathcal{A})_{\geq 4}$ *is algebraic.*

If $\mathbf{Si}(\mathcal{A})_{\geq 4}$ is finite, then F is a polynomial and so we obtain:

Corollary 13 *If the wreath-closed class $\mathcal{A}$ has only a finite number of simple permutations, then its generating function is algebraic.*

To prove Theorem 9 we have to consider subclasses of a wreath-closed class. These subclasses are defined by imposing further pattern restrictions. Therefore we shall need an analysis of sets of the form $\pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]$ where π is simple, and properties of their restrictions.

Lemma 14 *Suppose that $\pi \in \mathcal{S}_k$ is simple, $k \geq 4$. Then:*

$$\pi[\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k] \cap \pi[\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k] = \pi[\mathcal{A}_1 \cap \mathcal{B}_1, \mathcal{A}_2 \cap \mathcal{B}_2, \ldots, \mathcal{A}_k \cap \mathcal{B}_k].$$

This lemma follows directly from Proposition 2 and a similar result applies to $(12)[\mathcal{A}_1, \mathcal{A}_2] \cap (12)[\mathcal{B}_1, \mathcal{B}_2]$ (and to $(21)[\mathcal{A}_1, \mathcal{A}_2] \cap (21)[\mathcal{B}_1, \mathcal{B}_2]$) provided that $\mathcal{A}_1$ and $\mathcal{B}_1$ contain only plus-indecomposable (minus-indecomposable) permutations.

To prove a more powerful lemma about the restrictions of sets defined by inflating a permutation by some classes, we need two new definitions.

Definition 2 *Let $\mathcal{C}$ be a class of permutations. A* strong subclass*, $\mathcal{D}$, of $\mathcal{C}$ is a proper subclass of $\mathcal{C}$ which has the property that every basis element of $\mathcal{D}$ is involved in some basis element of $\mathcal{C}$.*

For example, the class whose basis consists of 231 is a strong subclass of the class whose basis consists of 2413 and 4231, since 231 is involved in 2413 (and it is a subclass because it is also involved in 4231). On the other hand, the class whose basis is 231 and 123, while still a subclass, is not a strong subclass of this class, since 123 is not involved in either 2413 or 4231. Since the basis of the intersection of two classes is a subset of the union of their bases it follows that the intersection of two strong subclasses of a class $\mathcal{C}$ is also a strong subclass of $\mathcal{C}$. Furthermore, since involvement is transitive, so is the strong subclass relation.

Definition 3 *Let γ and β be two permutations, and let the degree of β be n. An* embedding by blocks *of γ in β consists of a block decomposition $\gamma = \gamma_1\gamma_2\cdots\gamma_m$ whose pattern σ is a subpermutation of β together with a function $e : [m] \to [n]$ expressing the subpermutation embedding.*

For example, there are 7 embeddings by blocks of 213 into 3142; they arise from the block decompositions where

1. 213 is blocked as three singletons $2, 1, 3$ which map respectively to $3, 1, 4$
2. 213 is blocked as $21, 3$ and the two blocks map to $3, 4$ or to $1, 2$
3. 213 is a single block which the embedding maps to 3 or 1 or 4 or 2.

Lemma 15 *Suppose that $\pi \in \mathcal{S}_k$ is simple, $k \geq 4$, $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k$ are classes of permutations and $\gamma_1, \gamma_2, \ldots, \gamma_b$ is a sequence of permutations. Then:*

$$\pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]\langle\gamma_1, \gamma_2, \ldots, \gamma_b\rangle$$

can be represented as a union of sets of the form:

$$\pi[\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k]$$

where for $1 \leq i \leq k$, $\mathcal{V}_i$ is $\mathcal{W}_i\langle\gamma_1, \gamma_2, \ldots, \gamma_b\rangle$ or a strong subclass of this class.

Proof: It suffices to consider the case $b = 1, \gamma_1 = \gamma$ since the result then follows easily by induction. Let E be the set of all embeddings by blocks of γ in π.

We are interested in the permutations

$$\alpha = \alpha_1 \cdots \alpha_k \in \pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]$$

that do not involve γ. If γ were a subpermutation of some element $\alpha = \alpha_1 \cdots \alpha_k \in \pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]$, then there would be an embedding by blocks of γ in π such that each of the parts γ_i of the decomposition would be a subpermutation of $\alpha_{e(i)}$. So the elements of $\pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]\langle\gamma\rangle$ are those for which no $e \in E$ is such an embedding; hence for every $e \in E$ there is some part γ_i that is not a subpermutation of $\alpha_{e(i)}$. Therefore

$$\pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]\langle\gamma\rangle = \bigcap_{e\in E} \bigcup_i \pi[\mathcal{W}_1, \ldots, \mathcal{W}_{e(i)}\langle\gamma_i\rangle, \ldots, \mathcal{W}_k]. \qquad (2)$$

Using distributivity of intersection over union we may write the right hand side as a union of terms, each of which is an intersection of terms like

$$\pi[\mathcal{W}_1, \ldots, \mathcal{W}_j\langle\gamma_i\rangle, \ldots, \mathcal{W}_k].$$

These intersections, by Lemma 14, have the form $\pi[\mathcal{V}_1, \ldots, \mathcal{V}_k]$ where each $\mathcal{V}_j$ is either $\mathcal{W}_j$ or $\mathcal{W}_j$ restricted by finitely many permutations. In fact, because among the embedding by blocks of γ in π are all the embeddings which send γ into a single element of π, each $\mathcal{V}_j$ is of the form:

$$\mathcal{W}_j\langle\gamma, \ldots\rangle$$

where the permutations occurring after γ (if any) are blocks of γ and hence $\mathcal{V}_j$ is either $\mathcal{W}_j\langle\gamma\rangle$ or a strong subclass thereof as claimed. ■

As for Lemma 14 a similar result applies in the cases $\pi = 12, 21$ with appropriate indecomposability conditions.

We can make use of this lemma in enumerative situations. Namely, the size of

$$\mathcal{S}_n \cap \pi[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k]\langle\gamma_1, \gamma_2, \ldots, \gamma_b\rangle$$

can be computed from the sizes of the sets

$$\mathcal{S}_n \cap \pi[\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k]$$

and the sizes of their intersections using the principle of inclusion-exclusion. However, the intersection of any family of such sets is also such a set and so we see that the size of the original set is a combination with positive and negative coefficients of sizes of sets of the latter type.

A finitely based class has only finitely many strong subclasses since the closure downward of its basis under involvement is a finite set. So we may use the strong subclass relationship as a basis for inductive proofs. That is, if some property P holds of the class consisting only of the permutation 1, and if it is the case that, whenever all the strong subclasses of a class $\mathcal{C}$ satisfy P, then $\mathcal{C}$ satisfies P, then it follows that every finitely based class satisfies P.

We can now prove Theorem 9. The proof will be phrased as a proof by contradiction. However, this is simply a rhetorical device in order to avoid having to discuss detailed constructions. It will be important to note in the examples that it can be read effectively.

Proof: By Theorem 8 any class containing only finitely many simple permutations is defined by a finite set of restrictions. So, if the result were not true, we could find a class $\mathcal{C}$ for which it failed, but such that all the strong subclasses of $\mathcal{C}$ had algebraic generating functions. Let $\mathcal{W}$ be the wreath closure of $\mathbf{Si}(\mathcal{C})$, and let $\gamma_1, \gamma_2, \ldots, \gamma_b$ be a minimal sequence of permutations such that

$$\mathcal{C} = \mathcal{W}\langle \gamma_1, \gamma_2, \ldots, \gamma_b \rangle.$$

Note that $b \geq 1$ since Corollary 13 implies that the generating function of $\mathcal{W}$ is algebraic. Then, by Lemma 10, we also have a decomposition into disjoint sets:

$$\begin{aligned} \mathcal{C} &= \{1\} \cup (12)[\mathcal{W}_+, \mathcal{W}]\langle \gamma_1, \gamma_2, \ldots, \gamma_b \rangle \cup \\ & (21)[\mathcal{W}_-, \mathcal{W}]\langle \gamma_1, \gamma_2, \ldots, \gamma_b \rangle \cup \\ & \bigcup_{\pi \in \mathbf{Si}(\mathcal{C})_{\geq 4}} \pi[\mathcal{W}, \mathcal{W}, \ldots, \mathcal{W}]\langle \gamma_1, \gamma_2, \ldots, \gamma_b \rangle. \end{aligned}$$

Consider now any single set other than $\{1\}$ appearing on the right hand side of the expression defining $\mathcal{C}$. Using Lemma 15 and the observation about plus and minus decomposability following it, that set is the union of of sets of the form $\pi[\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k]$ where $k \geq 2$ and each $\mathcal{D}_i$ is either $\mathcal{C}$ or one of its strong subclasses. This union is not necessarily disjoint. However, the intersection of any two such sets is again a set of the same type, and since the generating function of $\pi[\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k]$ is simply equal to $D_1 D_2 \cdots D_k$ it follows, using the principal of inclusion/exclusion and then combining all the terms that result, that there is some polynomial p such that:

$$C = x + p(C, C_1, C_2, \ldots, C_m).$$

where $C_1, C_2, \ldots, C_m$ are the generating functions of all the strong subclasses of $\mathcal{C}$ and each term in p has degree at least two. This equation cannot be vacuous as all the generating functions involved have x as their term of lowest degree. Therefore the generating function of $\mathcal{C}$ is algebraic, providing the desired contradiction. ∎

6 Examples

In this section we consider a series of examples which apply (and in some cases extend) the results of the preceding sections. The first example is a simple illustration of the constructive nature of the proof of Theorem 9.

Example 1 *Let $\mathcal{W}$ be the wreath closure of the set of simple permutations $\{12, 21, 2413, 3142\}$, and let $\mathcal{C} = \mathcal{W}\langle 321\rangle$. Then the generating function of $\mathcal{C}$ is:*

$$C(x) = \frac{x(x^4 - x^3 + 4x^2 - 3x + 1)}{1 - 5x + 9x^2 - 8x^3 + 2x^4 - x^5}.$$

We begin by considering the embeddings by blocks of 321 into the simple members of $\mathcal{W}$. We may always embed with a singleton range, so we consider the remaining embeddings. For 12 there are no others. For 21 there are two, depending whether we send a single element or a pair to the first position. For 2413 and 3142 we have a richer collection of such embeddings, but they may all be described as sending a singleton or pair to the larger element of a descending pair, and the remainder of 321 to the smaller element. Since the parts of an inflation are non-empty, the restriction by 1 of such a part is empty. Furthermore the restriction by 21 of $\mathcal{W}$ is $\mathcal{I}$, the class of increasing permutations.

This simplifies the computations considerably. We obtain:

$$\begin{array}{rcl} (12)[\mathcal{W}_+, \mathcal{W}]\langle 321\rangle & = & (12)[\mathcal{W}_+\langle 321\rangle, \mathcal{W}\langle 321\rangle] \\ (21)[\mathcal{W}_-, \mathcal{W}]\langle 321\rangle & = & (21)[\mathcal{I}, \mathcal{I}] \\ (2413)[\mathcal{W}, \mathcal{W}, \mathcal{W}, \mathcal{W}]\langle 321\rangle & = & (2413)[\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I}] \\ (3142)[\mathcal{W}, \mathcal{W}, \mathcal{W}, \mathcal{W}]\langle 321\rangle & = & (3142)[\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I}] \end{array}$$

Now we can use this (and similar information about $\mathcal{W}_+$ derived in exactly the same way) in lemma 10 to obtain:

$$\begin{array}{rcl} C & = & x + C_+C + \dfrac{x^2}{(1-x)^2} + \dfrac{2x^4}{(1-x)^4} \\ C_+ & = & x + \dfrac{x^2}{(1-x)^2} + \dfrac{2x^4}{(1-x)^4} \end{array}$$

The terms on the right hand side of the equation for C arise directly from the preceding group of equations about sets of permutations together with the fact that the ordinary generating function for the class $\mathcal{I}$ is just $x/(1-x)$, while those for C_+ derive from the analogous information about $\mathcal{W}_+$.

The solution of this system is the generating function given above. Its series is:

$$C(x) = x + 2x^2 + 5x^3 + 14x^4 + 40x^5 + 111x^6 + 299x^7 + 793x^8 + \cdots$$

and the exponential constant governing the growth of the coefficients is approximately 2.6618.

Obviously the technique used here applies to the wreath closure of any finite set of simple permutations restricted by 321 (or of course 123). It can then be used inductively for any restriction of such a class by the identity permutation or its reverse. That is, we obtain:

Proposition 16 *Let $\mathcal{C}_n(\mathcal{F})$ be the class obtained by restricting the wreath closure of a finite set $\mathcal{F}$ of permutations by $n(n-1)\cdots 321$. Then $\mathcal{C}_n(\mathcal{F})$ has a rational generating function.*

Example 2 *Every proper subclass of $\mathcal{S}\langle 132\rangle$ has a rational generating function.*

In [11] it was shown that every class of the form $\mathcal{S}\langle 132, \tau\rangle$ where $132 \not\preceq \tau$ has a rational generating function. Using the proof Theorem 9 we can show that this same result holds for any proper subclass of $\mathcal{S}\langle 132\rangle$.

First consider $\mathcal{A} = \mathcal{S}\langle 132\rangle$ itself. As both simple permutations of length 4 involve 132, all simple permutations except 12 and 21 do. So we immediately obtain that all subclasses of $\mathcal{A}$ are finitely based (as $\mathcal{A}$ is a subclass of the class of separable permutations this was already established in [3]).

Although we cannot, in this case, apply Lemma 10 since $\mathcal{A}$ is not wreath-closed there is nevertheless an analogous structural result for $\mathcal{A}$. Namely:

$$\begin{aligned}
\mathcal{A} &= \{1\} \cup (12)[\mathcal{A}_+, \mathcal{I}] \cup (21)[\mathcal{A}_-, \mathcal{A}] \\
\mathcal{A}_+ &= \{1\} \cup (21)[\mathcal{A}_-, \mathcal{A}] \\
\mathcal{A}_- &= \{1\} \cup (12)[\mathcal{A}_+, \mathcal{I}]
\end{aligned}$$

These equations follow from the fact that a plus decomposition $(12)[\alpha, \beta]$ avoids 132 if and only if α avoids 132 and β is increasing, while a minus decomposition $(21)[\alpha, \beta]$ avoids 132 if and only if both α and β avoid 132.

Now consider $\mathcal{B}$ a proper subclass of $\mathcal{A}$ and choose a minimal sequence of permutations $\gamma_1, \gamma_2, \ldots, \gamma_b$ such that $\mathcal{B} = \mathcal{A}\langle\gamma_1, \gamma_2, \ldots, \gamma_b\rangle$. Suppose also that all strong subclasses of $\mathcal{B}$ have rational generating functions. We now can follow essentially the line of argument used in the proof of Theorem 9, making use in this case that each of the γ_i is either plus- or minus-decomposable.

If τ is a plus-decomposable permutation, then the set:

$$(12)[\mathcal{A}_+, \mathcal{I}]\langle\tau\rangle$$

will transform using (2) into a union of sets each of which is of the form $(12)[\mathcal{X}, \mathcal{Y}]$ where $\mathcal{X} = \mathcal{A}_+\langle\tau'\rangle$ for some τ' properly involved in τ, and $\mathcal{Y}$ is either $\mathcal{I}$ or some finite subclass of $\mathcal{I}$.

Replacing τ by each γ_i in turn we see that, if at least one γ_i is plus-decomposable, then the plus-decomposable elements of $\mathcal{B}$ will be a union of sets of the form $(12)[\mathcal{C}_+, \mathcal{D}]$ where $\mathcal{C}$ is some strong subclass of $\mathcal{B}$ and $\mathcal{D}$ is a subclass of $\mathcal{I}$. As these sets have rational generating functions and are closed under intersection, the plus-decomposable elements of $\mathcal{B}$ will have a rational generating function.

Likewise, if τ is minus-decomposable, we get a similar reduction of the minus-decomposables into sets of the form $(21)[\mathcal{C}_-, \mathcal{E}]$ where, as before $\mathcal{C}$ is a strong subclass of $\mathcal{B}$, but $\mathcal{E}$ is either $\mathcal{B}$ or one of its strong subclasses. So, if at least one γ_i is minus-decomposable, the minus-decomposable elements of $\mathcal{B}$ will have a rational generating function.

So either $\mathcal{B}_-$ or $\mathcal{B}_+$ must have a rational generating function, but it then follows immediately that $\mathcal{B}$ also does.

As noted following the proof of Theorem 9 this entire procedure is constructive. We have implemented the reductions it provides and as an example of the results which this code can produce we can show that the generating function for the class of permutations with basic restrictions $\{132, 34521, 43512\}$ is:

$$\frac{x(x^6 + 3x^5 + 2x^4 - 2x^3 - 4x^2 + 4x - 1)}{(1-x)^2(1-2x-x^2)^2}.$$

Example 3 *Every wreath-closed class all of whose simple permutations (apart from* $1, 12, 21$*) are exceptional is finitely based and has an algebraic generating function.*

The prime reason for giving this example is to show that we are not necessarily stymied if the number of simple permutations is infinite. The exceptional simple permutations fall into four infinite chains with four permutations of each even degree at least 6 and only two of length 4. So, in any class $\mathcal{A}$ whose simple permutations are all exceptional, the generating function of the simple permutations has the form

$$\frac{cx^4}{1-x^2} + p(x)$$

where $0 \leq c \leq 4$ and $p(x)$ is a polynomial. Consequently, if $\mathcal{A}$ is wreath-closed, its generating function is algebraic by Corollary 12.

Turning now to the basis of $\mathcal{A}$ we note first that every basis permutation is simple (Proposition 1). A basis permutation that was exceptional would belong to one of the 4 infinite chains discussed above and it is easy to see would have to be the smallest member in the chain that failed to belong to $\mathcal{A}$. So there cannot be more than 4 such. If β is a non-exceptional basis permutation, then, by Theorem 4, it would have a one-point deletion that was simple, necessarily in $\mathcal{A}$, and therefore exceptional. From now on we may assume that β is obtainable from an exceptional simple permutation σ by inserting a new value v somewhere within σ and relabelling appropriately.

Now we use two simplifying devices. The first is that we shall not, in fact, relabel the result of inserting v within σ; instead we shall regard v as being some non-integral value. The second is that, by an appropriate reversal or inversion if necessary, we can take σ to be $2\,4\,6\,\ldots\,(2m)\,1\,3\,5\,\ldots\,(2m-1)$ for some m. We therefore have

$$\beta = 2\,4\,6\,\cdots\,(2m)\,1\,3\,5\cdots(2i-1)\,v\,(2i+1)\,\cdots\,(2m-1)$$

The notation indicates that we are taking v in the second half of β but the first half can be handled in the same way. If $m > 2$, then either v is not adjacent to 1 or not adjacent to $2m-1$. In the former case we may remove the symbols 1 and 2 and obtain a simple permutation and in the latter case remove the symbols $2m-1$ and $2m$; but the resulting simple permutation is not exceptional, a contradiction. It follows that β has length at most 5.

Evidently, this argument is constructive and is capable of delivering the precise basis in any particular case. For example, if $\mathcal{A}$ is the wreath-closed class whose simple permutations are $1, 12, 21$ together with all the exceptional ones, the basis is the set of all six simple permutations of length 5.

7 Summary and conclusions

We have shown that an understanding of the simple permutations of a class can be very helpful in finding its generating function and its set of minimal pattern restrictions. In the case that the number of simple permutations is finite we have a complete answer to these problems. For wreath-closed classes we can often answer these questions also even if there are an infinite number of simple permutations. The outstanding open questions centre on subclasses of the wreath closure of an infinite number of simple permutations where, without Higman's theorem, we have no tool to prove these well quasi-ordered even if the simple permutations themselves are well quasi-ordered. It would be useful to resolve either way the question of whether there exists an infinite set of simple permutations whose wreath closure is well quasi-ordered. If such wreath closures existed, then we would be hopeful of adapting the techniques of Section 5 to obtain concrete information concerning their enumeration and general structure.

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