Partially well-ordered closed sets of permutations

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Abstract

It is known that the "pattern containment" order on permutations is not a partial well-order. Nevertheless, many naturally defined subsets of permutations *are* partially well-ordered, in which case they have a strong finite basis property. Several classes are proved to be partially well-ordered under pattern containment. Conversely, a number of new antichains are exhibited that give some insight as to where the boundary between partially well-ordered and not partially well-ordered classes lies.

Keywords Permutation, pattern containment, involvement, finite basis, partial well-order

1 Introduction

The relation of "pattern containment" or "involvement" on finite permutations has been studied in several recent papers. It arises in the context of sets of permutations being characterised by forbidden subpermutations. For example, in [16], permutations which are the union of an increasing sequence and a decreasing sequence are characterised by their avoiding 3412 and 2143; and, in [6], the class of separable permutations is characterised by their avoiding 2413 and 3142. Other papers, old and new, with similar characterisations are [13, 18, 10, 12]. The first characterisations of this type (of stack sortable and restricted deque sortable permutations) go back to [9].

Formally, one permutation $\sigma = s_1, \ldots, s_m$ is said to be *involved* in another permutation $\tau = t_1, \ldots, t_n$ when $t_1, \ldots, t_n$ has a subsequence that is order isomorphic to $s_1, \ldots, s_m$. We write $\sigma \preceq \tau$ to express this. Those sets X of permutations defined by their avoiding a set of forbidden patterns are precisely the *closed* sets defined in [1]: they satisfy the condition

$$\tau \in X \text{ and } \sigma \preceq \tau \Rightarrow \sigma \in X$$

Every closed set X is defined by a minimal set of forbidden permutations (namely, those permutations not in X all of whose proper subpermutations belong to X); this (unique) minimal set is called the *basis* of X. Notice that every basis forms an antichain in the involvement order and that every antichain is the basis of a unique closed set. We denote the closed set that has basis $\{\beta_1, \beta_2, \ldots\}$ by $A(\beta_1, \beta_2, \ldots)$.

It has long been known that infinite antichains exist (see [13, 18]); in other words, not every closed set is finitely based. On the other hand, many of the naturally arising closed sets are finitely based and therefore it seems to be a significant problem to give conditions under which a closed set would have a finite basis.

In this paper we study an even stronger property of closed sets. We say that a closed set X is *strongly finitely based* if all its closed subsets are finitely based. Of course, such a notion can be defined for any partial order and, according to Higman [8], goes back to Erdös and Rado. Similarly, the following result is not new either but we know of no convenient reference and, for completeness, we give the proof in the present situation.

Proposition 1.1 *Let X be a finitely based closed set of permutations. Then the following are equivalent:*

1. *Every closed subset of X is finitely based (X is strongly finitely based),*
2. *X has at most countably many closed subsets,*
3. *X has no infinite antichain,*
4. *The closed subsets of X satisfy the minimum condition under inclusion.*

PROOF: $(1 \Rightarrow 2)$ Obvious as there are only countably many possible bases.

$(2 \Rightarrow 3)$ Suppose T was an infinite antichain in X. For every $S \subseteq T$ define $\mathrm{cl}(S) = \{\pi | \pi \preceq \sigma, \sigma \in S\}$ to be the closed set generated by S. In $\mathrm{cl}(S)$ the elements of S are all maximal and are the only maximal elements. Thus $S_1 \neq S_2$ implies $\mathrm{cl}(S_1) \neq \mathrm{cl}(S_2)$; so there are uncountably many closed subsets of X.

$(3 \Rightarrow 4)$ Suppose that there exists a family of closed subsets of X with no minimal element. Then, inductively, we can find an infinite properly descending chain

$$A_1 \supset A_2 \supset \ldots$$

of closed subsets of X. Since the inclusions are proper we can choose permutations $\alpha_i \in A_i \setminus A_{i+1}$. The set of minimal elements of $\{\alpha_1, \alpha_2, \ldots\}$ is an antichain and is therefore finite; so, for some finite n, $\{\alpha_1, \ldots, \alpha_n\}$ contains all these minimal elements. Then a_{n+1} involves some a_m with $m \leq n$. Since A_{n+1} is closed, $a_m \in A_{n+1}$, a contradiction.

(4 $\Rightarrow$ 1) Let Y be a closed subset of X and let

$$\mathcal{Z} = \{Z \mid Y \subseteq Z \subseteq X, Z \text{ closed and finitely based}\}$$

Since $X \in \mathcal{Z}$, $\mathcal{Z}$ is not empty. Let W be a minimal member of $\mathcal{Z}$ (under set inclusion) having the finite set B as its basis. If Y was a proper subset of W we could choose some $\omega \in W \setminus Y$ and have

$$Y \subseteq A(B) \cap A(\{\omega\}) = A(B \cup \{\omega\}) \subset W$$

which would contradict the minimality of W. Therefore $Y = W$ is finitely based. ■

In general, a partially ordered set is said to be *partially well-ordered* if it has no infinite properly descending chain, nor infinite antichain. In our situation, since all permutations are finite, there certainly cannot be any infinite descending chain and so we shall simply say that a set of permutations is *partially well-ordered* precisely when it contains no infinite antichain.

We note, by the way, that closed sets which have only *boundedly* finite antichains are characterised by Theorem 1 of [5]. However, most strongly finitely based sets will have antichains of unbounded length. Nevertheless, one would expect that the property of being strongly finitely based would be enjoyed only by closed sets which are 'small' in some sense. This paper gives a number of positive and negative results to provide some idea of where the boundary between strongly finitely based and non-strongly finitely based sets lies.

In the next section we gather together some technical prerequisites and introduce the main theoretical aids for our results. This section also contains results which bear on the strong finite basis condition in general. Section 3 is devoted to the description of three infinite antichains which we use in the subsequent sections. In Section 4 we consider closed sets with a basis of two permutations of lengths 3 and 4. We decide the strong finite basis question in all such cases. Next we go on to study a two-parameter family of closed sets $B(a, b)$ defined by an inductive property: permutations of length n in $B(a, b)$ are formed by inserting the element n either among the first a positions or the last b positions of a permutation in $B(a, b)$ of length $n - 1$. In Section 5 we determine the set of all (a, b) for which $B(a, b)$ is strongly finitely based. Finally we consider a number of closed sets which have arisen in various different areas of combinatorics and computer science and observe that they are not partially well-ordered.

2 Groundwork

We begin by noting the following result whose proof follows easily from Proposition 1.1 and Theorem 2.1 of [1].

Lemma 2.1 *The union of a finite number of strongly finitely based closed sets of permutations is strongly finitely based.*

The above lemma has the following strong converse.

Theorem 2.2 *Every strongly finitely based closed set X can be represented as a union of a finite number of closed sets*

$$X = X_1 \cup X_2 \cup \ldots \cup X_k$$

where no X_i is further expressible as a union of proper closed subsets. Furthermore, if no X_i is redundant ($X_i \not\subseteq \cup_{j \neq i} X_j$) then $X_1, \ldots, X_k$ are unique.

PROOF: If the theorem were false then the set of closed subsets of X which were not finite unions would have a minimal element and a contradiction would easily follow. Uniqueness follows by a standard argument. ■

In [11] closed sets which are not the unions of proper closed subsets are studied in detail and a structure theorem for them is given.

In order to obtain deeper results we generally resort to a famous theorem of Higman [8] that applies to algebras endowed with operations which respect various order conditions. In most of our applications we use a special case where only one operation is present and for clarity we shall state this case explicitly. At one point though we need the theorem for an algebra with two operations and then we ask the reader to refer directly to [8].

Let Σ be a set endowed with a partial order $\leq$ and let Σ^+ be the set of all non-empty words over the alphabet Σ. We extend the partial order to the *dominance* order on Σ^+ by the rule

$$s_1 \ldots s_m \leq t_1 \ldots t_n$$

if and only if, for some $1 \leq i_1 < \ldots < i_m \leq n$ we have

$$s_j \leq t_{i_j}$$

Theorem 2.3 *(Higman) If Σ is partially well-ordered then Σ^+ is partially well-ordered by the inherited dominance order.*

We use this theorem in many different ways the simplest of which is the case that Σ is finite and the only comparabilites are the trivial ones $s \leq s$; in that case the theorem says that Σ^+ is partially well-ordered by the subsequence ordering.

Before giving our first application we introduce some terminology that will be used throughout the paper. The *sum* of two permutations $\sigma = s_1, \ldots, s_m$ and $\tau = t_1, \ldots, t_n$ is the permutation $\sigma \oplus \tau = x_1, \ldots, x_{m+n}$ where $x_i = s_i$ if $1 \leq i \leq m$ and $x_{i+m} = t_i + m$ if $1 \leq i \leq n$. This simply means that the first m symbols are permuted in the same way as σ and the remaining n symbols are permuted in the same way as τ suitably translated by the addition of m. We extend this definition to closed sets X and Y by defining $X \oplus Y$ as the set of all permutations $\theta \oplus \phi$ where $\theta \in X$ and $\phi \in Y$. Clearly $X \oplus Y$ is also closed but, even if X and Y are finitely based, it need not be finitely based [2].

Notice that every permutation has a unique decomposition as $\sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_k$ into component permutations which cannot further be decomposed. If $k = 1$ we say that σ is an *indecomposable* permutation.

Also, if σ and τ are sequences and $s < t$ for every $s \in \sigma$ and $t \in \tau$ then we write $\sigma < \tau$. In particular we use this notation when either of σ and τ is an integer (regarded as a sequence of length 1).

There is similar operation on permutations, called the *skew sum*. The permutation $\sigma \ominus \tau$ denotes the unique permutation whose first m symbols are permuted in the same way as σ, all of them being greater than the last n symbols, and the last n symbols being permuted in the same way as τ. For example $54231 = 21 \ominus 231$. Obviously $X \ominus Y = \{\sigma \ominus \tau \mid \sigma \in X, \tau \in Y\}$ is closed if X and Y are closed.

Lemma 2.4 *If X and Y are partially well-ordered closed sets then both $X \oplus Y$ and $X \ominus Y$ are partially well-ordered.*

PROOF: Let $A = \{\theta_i \oplus \phi_i \mid i = 1, 2, \ldots\}$ be any infinite subset of $X \oplus Y$. Then $\{(\theta_i, \phi_i) \mid i = 1, 2, \ldots\}$ is an infinite subset of the direct product $X \times Y$. Since $X \times Y$ is partially well-ordered there exist two pairs (θ_i, ϕ_i) and (θ_j, ϕ_j) with $\theta_i \preceq \theta_j$ and $\phi_i \preceq \phi_j$. It follows that $\theta_i \oplus \phi_i \preceq \theta_j \oplus \phi_j$ and therefore A is not an antichain. Therefore, from Proposition 1.1, $X \oplus Y$ is partially well-ordered. The result for $X \ominus Y$ is proved in the same way. ∎

Our second application is also quite straightforward but it has a consequence that is not easily proved by other means. For any set X of permutations we define, respectively, $S \wr X$ and its subset $I \wr X$ to be the set of permutations generated by X under, respectively, both the operations $\oplus$ and $\ominus$, and only the operation $\oplus$. (This odd notation is for consistency with [2]). It is easy to verify that, ordered by involvement, $S \wr X$ satisfies the conditions of Higman's theorem with respect to these two operations. Therefore we have

Theorem 2.5 *If X is partially well-ordered then $S \wr X$ and $I \wr X$ are partially well-ordered.*

Corollary 2.6 *The closed set $A(3142, 2413)$ is strongly finitely based.*

PROOF: If we take $X = \{1\}$ then $S \wr X$ is the class of separable permutations defined in [6], and there it is proved that $\{3142, 2413\}$ is its basis. ∎

Corollary 2.7 *The closed set $A(231)$ is strongly finitely based.*

PROOF: Clearly, $A(231) \subseteq A(3142, 2413)$. ∎

This last result is in sharp contrast to the result of [15] where an infinite antichain in $A(123)$ is constructed. Thus, despite $A(231)$ and $A(123)$ being equinumerous they are completely different as partially ordered sets. Note that every permutation of length 3 is equivalent under the standard symmetries generated by reversal, complementation, and inversion (see [14]) to one of 231 and 123.

Another useful family of closed sets is the "generalised W's". Let $\omega = w_1 \dots w_k$ be a finite sequence of ± 1's. Then $W(\omega)$ is the set of all permutations $\alpha_1 \dots \alpha_k$ where, for each i, α_i is an ascending sequence if $\omega_i = +1$, or a descending sequence if $\omega_i = -1$ (and, in either case, may be empty).

As an example, consider the permutation 258976134. The segments 258, 976, 134 witness that the permutation lies in $W(+1,-1,+1)$ but they are not unique witnesses.

It can sometimes be technically troublesome that the decomposition that witnesses that a permutation belongs to $W(\omega)$ is not unique. To overcome this we shall suppose that a fixed decomposition is chosen once and for all for each $\sigma \in W(\omega)$. Given this choice we define another relation on $W(\omega)$. If $\alpha_1 \dots \alpha_k$ and $\beta_1 \dots \beta_k$ are two permutations of $W(\omega)$ with decompositions as shown then we say

$$\alpha_1 \dots \alpha_k \preceq' \beta_1 \dots \beta_k$$

if each β_i has a subsequence β_i' where $|\alpha_i| = |\beta_i'|$ and $\alpha_1 \dots \alpha_k$ is order isomorphic to $\beta_1' \dots \beta_k'$. This new relation is also a partial order and it is a stricter one than involvement. Of course, the new relation depends on the initial choice of decompositions of permutations in $W(\omega)$ although that dependence is not made explicit in the notation.

Lemma 2.8 *$W(\omega)$ is partially well-ordered by the $\preceq'$ relation.*

PROOF: Let $\Sigma = \{1, 2, \dots, k\}$. We encode every $\sigma = \alpha_1\alpha_2 \dots \alpha_k \in W(\omega)$ of length n as a word $\chi(\sigma) = c_1c_2 \dots c_n \in \Sigma^+$ by defining $c_i = \ell$ if and only if $i \in \alpha_\ell$. Note that σ can be recovered from $\chi(\sigma)$ since $\chi(\sigma)$ determines the set of symbols that comprise each α_ℓ, and these symbols will occur in increasing or decreasing order according as $\omega_\ell = +1$ or $\omega_\ell = -1$.

Suppose that $\sigma, \tau \in W(\omega)$ and that $\chi(\sigma)$ is a subsequence of $\chi(\tau)$. We aim to prove that $\sigma \preceq' \tau$. Put $\chi(\sigma) = c_1 \dots c_m$ and $\chi(\tau) = d_1 \dots d_n$ and suppose that $c_1 \dots c_m = d_{i_1} \dots d_{i_m}$. Consider the set $\{i_1, \dots, i_m\}$ and its arrangement within τ. As explained above, this arrangement within τ is determined by $d_{i_1} \dots d_{i_m}$ (and by ω) as $\beta_1' \dots \beta_k'$, say, where each β_i' is a subsequence of β_i. However, as $c_j = d_{i_j}$ for $1 \leq j \leq m$ and $c_1 \dots c_m$ determines $\alpha_1 \dots \alpha_k = \sigma$ we have that $|\alpha_i| = |\beta_i'|$ and that $\alpha_1 \dots \alpha_k$ is order isomorphic to $\beta_1' \dots \beta_k'$. Therefore $\sigma \preceq' \tau$ as required.

Now let A be an arbitrary infinite subset of $W(\omega)$. Then $\chi(A)$ is an infinite subset of Σ^+ which is partially well-ordered by the subsequence relation. Therefore we can find elements $\chi(\sigma), \chi(\tau) \in \chi(A)$ with $\chi(\sigma)$ a subsequence of $\chi(\tau)$. We deduce that $\sigma \preceq' \tau$ and therefore that A is not an antichain. This completes the proof. ■

Theorem 2.9 *Every $W(\omega)$ is strongly finitely based.*

PROOF: The involvement relation contains the relation $\preceq'$ and so, by the previous lemma, $W(\omega)$ is partially well-ordered under involvement. Theorem 2.2 of [1] tells us that $W(\omega)$ is finitely based and the proof is completed by appealing to Proposition 1.1. ■

Corollary 2.10 *The profile classes of [1] are strongly finitely based.*

PROOF: Every profile class is a subset of some $W(+1, +1, \ldots, +1)$. ■

Corollary 2.11 *If λ, μ are any two distinct permutations of length 3 then $A(\lambda, \mu)$ is strongly finitely based.*

PROOF: An examination of the structure of these sets (refining the study in [14]) shows that they are all covered by special cases of the results proved already. We shall be proving stronger results below from which the corollary can also be derived. ■

3 Some infinite antichains

In this section we construct three infinite antichains U, V, W which we shall use subsequently to show that various closed sets are not partially well-ordered.
The set U

$$\begin{aligned}
U_1 &= 2,3,5,1 \mid 6,7,4 \\
U_2 &= 2,3,5,1 \mid 7,4 \mid 8,9,6 \\
U_3 &= 2,3,5,1 \mid 7,4,9,6 \mid 10,11,8 \\
&\ldots \\
U_k &= 2,3,5,1 \mid 7,4,9,6,11,8,\ldots,2k+3,2k \mid 2k+4,2k+5,2k+2 \\
&\ldots
\end{aligned}$$

In U_k we have an initial segment $2, 3, 5, 1$ and a final segment $2k + 4, 2k + 5, 2k + 2$. In between these segments we have $7, 9, 11, 13, \ldots$ interleaved with $4, 6, 8, 10, \ldots$.
The set V

$$\begin{aligned}
V_1 &= 5,8 \mid 2,1,4 \mid 6,3 \mid 9,10,7 \\
V_2 &= 9,12,5,8 \mid 2,1,4 \mid 6,3,10,7 \mid 13,14,11 \\
V_3 &= 13,16,9,12,5,8 \mid 2,1,4 \mid 6,3,10,7,14,11 \mid 17,18,15 \\
&\ldots \\
V_k &= 4k+1,4k+4,4k-3,4k,\ldots,5,8 \mid 2,1,4 \mid 6,3,10,7,\ldots \\
&\quad\; 4k+2,4k-1 \mid 4k+5,4k+6,4k+3 \\
&\ldots
\end{aligned}$$

V_k has four parts. The first part is an interleaving of $4k + 1, 4k - 3, 4k - 7, \ldots, 9, 5$ with $4k + 4, 4k, 4k - 4, \ldots, 12, 8$. The second part is just $2, 1, 4$. The third part is an interleaving of $6, 10, 14, \ldots$ with $3, 7, 11, \ldots$, and the fourth part is $4k + 5, 4k + 6, 4k + 3$.
The set W

$$
\begin{aligned}
W_1 &= 8,1 \mid 5,3,6,7,9,4 \mid 10,11,2 \\
W_2 &= 12,1,10,3 \mid 7,5,8,9,11,6 \mid 13,4 \mid 14,15,2 \\
W_3 &= 16,1,14,3,12,5 \mid 9,7,10,11,13,8 \mid 15,6,17,4 \mid 18,19,2 \\
W_4 &= 20,1,18,3,16,5,14,7 \mid 11,9,12,13,15,10 \mid 17,8,19,6,21,4 \mid 22,23,2 \\
&\dots \\
W_k &= 4k+4,1,4k+2,3,\dots,2k+6,2k-1 \mid 2k+3,2k+1,2k+4, \\
&\quad 2k+5,2k+7,2k+2 \mid 2k+9,2k,\dots,4k+5,4 \mid 4k+6,4k+7,2 \\
&\dots
\end{aligned}
$$

W_k has a central segment of six terms. Preceding this are $2k$ terms that are an interleaving of $4k+4, 4k+2, 4k, \dots, 2k+6$ with $1, 3, 5, \dots, 2k-1$. Following the central section is an interleaving of $2k+9, 2k+11, \dots, 4k+3$ with $2k, 2k-2, \dots, 4$ and then, finally, the three terms $4k+6, 4k+7, 2$.

In these permutations the $\mid$ symbol is used only to clarify the structure of the permutations.

Proposition 3.1 *U, V and W are antichains.*

PROOF: We first consider the set U.

Let U_i' denote the permutation obtained by removing the largest term (namely $2i+5$) from U_i. Notice that the subsequence obtained by removing $2i+4$ from U_i is order isomorphic to U_i' and so U_i has at least two subsequences that are order isomorphic to U_i'. For brevity we say that U_i' has two *embeddings* in U_i.

Next we show that, for every $i < j$, U_i' has a unique embedding in U_j and that this consists of the first $2i+4$ terms of U_j. We prove this by induction on i.

To demonstrate that U_1' has a unique embedding in each U_j where $j > 1$, note that every permutation U_j has precisely two embeddings of 2341, these being the first four and the greatest four terms of that permutation. It is then easy to see that every U_j with $j > 1$ has a unique embedding of 23514 and of 235164. The latter is precisely U_1' and the unique embedding consists of the first six terms of U_j.

Now consider any U_i', U_j with $1 < i < j$. By induction we may assume that there is a unique embedding of U_{i-1}' in U_j and that this consists of the first $2i+2$ terms of U_j. The first $2i+2$ terms of U_j are as follows:

$$2,3,5,1,7,4,\dots,2i-1,2i-4,2i+1,2i-2,2i+3,2i$$

Note that there is only one term less than $2i+3$ that does not lie in this initial segment of U_j, and that is $2i+2$. Thus there exists a unique embedding in U_j of:

$$2,3,5,1,7,4,\dots,2i-1,2i-4,2i+1,2i-2,2i+3,2i,2i+2$$

As $2i+2$ is the $2i+4$th term of U_j we may conclude that there is a unique subsequence of U_j order isomorphic to the first $2i+4$ terms of U_j. The first $2i+4$ terms of U_j are:

$$2, 3, 5, 1, 7, 4, \ldots, 2i-1, 2i-4, 2i+1, 2i-2, 2i+3, 2i, 2i+5, 2i+2$$

But these are order isomorphic to U_i'. Thus there is a unique embedding of U_i' in U_j consisting of the first $2i+4$ terms of U_j. This completes our induction.

It is now clear that U is an antichain. For if we had $U_i \preceq U_j$ then we would have $i < j$; but the two embeddings of U_i' in U_i would give rise to two embeddings of U_i' in U_j which is impossible.

To demonstrate that V is an antichain an identical argument can be used. Again, let V_i' be the permutation obtained by removing the largest term from V_i. As before, there are two embeddings of V_i' in V_i, obtained by omitting the largest term or the second largest term.

Every V_j has a unique embedding of 21453, consisting of the subsequence 21463. From this we can deduce by induction that for all U_i and U_j with $i < j$ there is a unique embedding of U_i' in U_j, consisting of all the terms of U_j except for the first $2(j-i)$ and the last $2(j-i)+1$ terms. Thus we conclude that V is an antichain.

For W the argument is again the same. W_i' is defined to be the permutation obtained from W_i by removing the largest term. Each W_i has two distinct embeddings of W_i'. By examining subsequences order isomorphic to 3142 we can prove that each W_i has a unique embedding of 314562 consisting of the $2i+1$th to the $2i+6$th terms inclusive of W_i. Thus we demonstrate by induction that for every W_i, W_j with $i < j$ there is a unique embedding of W_i' in W_j consisting of all the terms of W_j except the first $2(i-j)$ terms and the last $2(i-j)+1$ terms. Thus we have that W is an antichain.

■

4 Basis permutations of lengths 3 and 4

In this section we consider closed sets of the type $A(\alpha, \beta)$ where the lengths of α and β are 3 and 4 respectively and $\alpha \not\preceq \beta$. Under the usual symmetry operations there are 18 inequivalent such sets (see [19, 1]). In 10 of them α is (equivalent to) 231 and so $A(\alpha, \beta) \subseteq A(231)$ is partially well-ordered and therefore strongly finitely based. The remaining 8 closed sets are represented by $A(321, \beta)$ where β is one of $1234, 2134, 1324, 2143, 3124, 2413, 3412, 4123$. In a series of lemmas we shall determine whether these sets are strongly finitely based.

Lemma 4.1 *1. $A(321, 1234)$ is a finite set.*

2. $A(321, 2134)$ is a subset of a profile class.

3. $A(321, 1324)$ is a profile class.
In particular, all these sets are strongly finitely based.

PROOF: That $A(321, 1234)$ is finite is a special case of a famous theorem of Erdös and Szekeres (see [7]). The other two parts follow from Propositions 3.1 and 3.2 of [1]. ■

Lemma 4.2 $A(321, 2143)$ *is strongly finitely based.*

PROOF: We appeal to Proposition 3.4 of [1]. This proposition, in the notation of the present paper, states that $A(321, 2143)$ is the union of $W(+1, +1)$ and the inverse of this set. The lemma then follows from Theorem 2.9 and Lemma 2.1. ■

Lemma 4.3 $A(321, 3124)$ *is strongly finitely based.*

PROOF: If $\sigma \in A(321, 3124)$ is written as $\sigma = \alpha n \beta$, where n is the maximal symbol of σ. Then

1. β is increasing and
2. α avoids 321 and 312

It is easily checked that the second condition is equivalent to α being a sum of cycles of the form $a+1, a+2, \ldots, a+t-1, a$ where, if $t = 1$, the cycle is called trivial. If α has only trivial cycles then $\sigma \in W(+1, +1)$. On the other hand, if α has a non-trivial cycle, let $C = a_2, a_3, \ldots, a_t, a_1$ be the rightmost such and write $\alpha = \alpha_1 \oplus \alpha_2$ where α_2 begins with the cycle C and is followed by an increasing sequence. Now, since σ avoids 321, we have $a_1 < \beta$. We conclude that $\sigma = \alpha_1 \oplus \rho$ where $\rho = \rho_1 \rho_2 \rho_3$ and each ρ_i is increasing.

This proves that $A(321, 3124) \subseteq A(321, 312) \oplus W(+1, +1, +1)$ and the lemma follows from the results in Section 2. ■

Lemma 4.4 $A(321, 2413)$ *is strongly finitely based.*

PROOF: Let D be the set of indecomposable permutations of $A(321, 2413)$. Suppose that $\theta \in D$ and $|\theta| = n$. Let m be minimal subject to θ having the form

$$\theta = \theta_m m \theta_{m+1} m+1 \ldots n \theta_n$$

where $\theta_m, \ldots, \theta_n$ are (possibly empty) segments of θ. By the minimality of m, $m-1 \notin \theta_m$.

We shall show that θ_m is a permutation of $1, 2, \ldots, t$ for some t. If that is not the case then we have $k \notin \theta_m$, $k+1 \in \theta_m$ for some k. But then, according to whether $m-1$ precedes k in θ, we have either a subsequence $m, m-1, k$ or a subsequence $k+1, m, k, m-1$. However, these subsequences are order isomorphic to 321 or 2413 respectively which is impossible.

It now follows that θ_m is empty since $\theta_m \oplus m\theta_{m+1} m+1 \ldots n\theta_n$ is a decomposition of θ.

Next notice that $\theta_{m+1}\theta_{m+2} \ldots \theta_n$ is an increasing sequence because any decreasing subsequence of length 2 would give, with m, a subsequence order isomorphic to 321. It follows that $\theta_{m+1}\theta_{m+2} \ldots \theta_n = 12 \ldots m-1$. It follows that $\theta^{-1} = \alpha\beta$ where α and β are increasing segments of length $m-1$ and $n-m+1$.

We have proved that $D^{-1} \subseteq W(+1,+1)$ and so, by Theorem 2.9, D is partially well-ordered by involvement. Finally, since $A(321, 2413) = D \wr I$, $A(321, 2413)$ is also partially well-ordered by involvement and therefore is strongly finitely based. ■

Lemma 4.5 *$A(321, 3412)$ and $A(321, 4123)$ are not strongly finitely based.*

PROOF: It is easily checked that both these closed sets contain the infinite antichain U. Notice that this gives the stronger result that $A(321, 3412, 4123)$ is not strongly finitely based. ■

We can summarise all the results above as

Theorem 4.6 *Suppose that α and β are permutations of lengths 3 and 4 and that $A(\alpha, \beta)$ is not strongly finitely based. Then the pair (α, β) is equivalent under symmetry to $(321, \gamma)$ where either $321 \preceq \gamma$ (in which case $A(321, \gamma) = A(321)$) or $\gamma = 3412$ or $\gamma = 4123$.*

5 The classes $B(a, b)$

The previous sections have studied the strong finite basis property for some closed sets given by a basis consisting of permutations of small length. This section considers closed sets $B(a, b)$ which are given in another way. Their definition is recursive: $\sigma \in B(a, b)$ if $\sigma = \alpha n \beta$ where n is the largest symbol of σ, $\alpha\beta \in B(a, b)$, and either $|\alpha| < a$ or $|\beta| < b$. The aim of this section is to prove the following theorem.

Theorem 5.1 *The set $B(a, b)$ is strongly finitely based if and only if (a, b) is one of the following pairs:*

$$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1).$$

We begin by noting two simple facts: $B(a, b)$ is equivalent under the reversal symmetry to $B(b, a)$ and $B(a, b) \subseteq B(a, b+1)$. In consequence the theorem will follow from the next three lemmas.

Lemma 5.2 *$B(0, 3)$ is not strongly finitely based.*

PROOF: It is easy to verify that the infinite antichain U is a subset of $B(0, 3)$. ■

Lemma 5.3 *$B(2, 2)$ is not strongly finitely based.*

PROOF: Here we verify that $V \subseteq B(2, 2)$. ■

Lemma 5.4 *$B(1, 2)$ is strongly finitely based.*

PROOF: Let $\sigma \in B(1,2)$. Then σ is constructed by inserting the symbols $2,3,\ldots$ in turn starting from the sequence 1. Each of the insertions places the current symbol either at the start of the sequence or in one of the two final places. Therefore we have $\sigma = \alpha\beta$ where α is the segment that has been contructed (from right to left) by insertion into the first place and β is the segment that has been constructed (from left to right) by inserting the current symbol into one of the last two places. It follows that $\alpha \in B(1,0)$ and so is decreasing, while $\beta \in B(0,2) = A(312,321)$ is the sum of "cycles", where a cycle is a sequence order isomorphic to a permutation of the form $a+1, a+2, \ldots, a+t-1, a$ for some $t \geq 1$.

By an appropriate division of α into segments we can write

$$\sigma = \lambda_k \ldots \lambda_2 \lambda_1 \rho_1 \rho_2 \ldots \rho_k$$

where each λ_i is decreasing, each ρ_i is a cycle, and $\lambda_i\rho_i < \lambda_{i+1}\rho_{i+1}$.

Let Σ be the set of all permutations of the form $\lambda\rho$ where λ is decreasing and ρ is a cycle. Evidently we have $\lambda\rho = \lambda\rho'\rho''$ where ρ' is increasing and $|\rho''| = 0$ or 1 and this representation proves that $\lambda\rho \in W(-1,+1,+1)$. Therefore, by Lemma 2.8, Σ is partially well-ordered under the relation $\preceq'$ defined in Section 2. It follows that Σ is also partially well-ordered under the order $\preceq''$ defined by $\lambda_1\rho_1 \preceq'' \lambda_2\rho_2$ if λ_2 and ρ_2 have subsequences λ_2^* and ρ_2^* where $|\lambda_1| = |\rho_1^*|$, $|\lambda_2| = |\rho_2^*|$ and $\lambda_1\rho_1$ is order isomorphic to $\lambda_2^*\rho_2^*$.

Then by Theorem 2.3 Σ^+ is partially well-ordered under the dominance order induced by $\preceq''$.

Now, using the form of the permutations $\sigma \in B(1,2)$ given above, we can encode every such permutation as a word

$$\chi(\sigma) = (\lambda_1\rho_1)(\lambda_2\rho_2)\ldots(\lambda_k\rho_k)$$

of Σ^+.

Suppose that A is an infinite subset of $B(1,2)$. Then $\chi(A)$ has two comparable elements $(\lambda_1\rho_1)(\lambda_2\rho_2)\ldots(\lambda_k\rho_k)$ and $(\lambda_1'\rho_1')(\lambda_2'\rho_2')\ldots(\lambda_\ell'\rho_\ell')$. So there exist $1 \leq j_1 < j_2 < \ldots < j_k \leq \ell$ such that $\lambda_i\rho_i \preceq'' \lambda_{j_i}'\rho_{j_i}'$ for each $i = 1,\ldots,k$. Since $\lambda_1\rho_1 < \lambda_2\rho_2 < \ldots < \lambda_k\rho_k$ and $\lambda_1'\rho_1' < \lambda_2'\rho_2' < \ldots < \lambda_\ell'\rho_\ell'$ it now follows that the corresponding permutations $\sigma = \lambda_k \ldots \lambda_1\rho_1 \ldots \rho_k$ and $\tau = \lambda_\ell' \ldots \lambda_1'\rho_1' \ldots \rho_\ell'$ satisfy $\sigma \preceq \tau$. ■

We end this section with a remark. It follows from the definition of $B(a,b)$ that this set is the juxtaposition (in the sense of [1]) of $B(a,0)$ and $B(0,b)$ both of which are easily seen to be finitely based. Therefore, by Theorem 2.2 of [1], all the sets $B(a,b)$ have a finite basis.

6 Concluding remarks

The antichains U, V, W suffice to prove that no closed set of the form $A(\sigma)$ with σ of length 4 is strongly finitely based. Indeed they can often prove rather more and we list some examples in the following theorem.

Theorem 6.1 *The following closed sets are not strongly finitely based:*

1. *The set of input restricted deque sortable permutations (see [9]),*
2. *The set of permutations which are the union of an increasing and a decreasing subsequence (see [16]),*
3. *The set of* smooth *permutations (see [10]),*
4. *The set of* $(2, 1)$ *stack sortable permutations (see [4]).*

PROOF:

1. This set has basis $\{4231, 3241\}$ and we can appeal to antichain U.
2. Here the basis is $\{3412, 2143\}$ and antichain W can be used.
3. The set of smooth permutations has basis $\{3412, 4231\}$ and therefore contains U.
4. The basis of this set is $\{2341, 3241\}$ and we use the antichain of inverses of the permutations of U.

■

References

[1] M. D. Atkinson: Restricted permutations, Discrete Math. 195 (1999), 27–38.

[2] M. D. Atkinson, T. Stitt: Restricted permutations and the wreath product, In preparation.

[3] M. D. Atkinson: Permutations which are the union of an increasing and a decreasing sequence, Electronic J. Combinatorics 5 (1998), Paper R6 (13 pp.).

[4] M. D. Atkinson: Generalised stack permutations, Combinatorics, Probability and Computing 7 (1998), 239–246.

[5] M. D. Atkinson, R. Beals: Finiteness conditions on closed classes of permutations, unpublished.

[6] P. Bose, J. F. Buss, A. Lubiw: Pattern matching for permutations, Inform. Process. Lett. 65 (1998), 277–283.

[7] D. I. A. Cohen: *Basic Techniques of Combinatorial Theory*, John Wiley & Sons, New York, 1978.

[8] G. Higman: Ordering by divisibility in abstract algebras, Proc. London Math. Soc. 2 (1952), 326–336.

[9] D.E. Knuth: *Fundamental Algorithms, The Art of Computer Programming* Vol. 1 (First Edition), Addison-Wesley, Reading, Mass. (1967).

[10] V. Lakshmibai, B. Sandhya: Criterion for smoothness of Schubert varieties, Proc. Indian Acad. Sci. Math. Sci. 100 (1990), no. 1, 45–52.

[11] M. M. Murphy: Ph.D. thesis, University of St Andrews, in preparation.

[12] L. Shapiro, A. B. Stephens: Bootstrap percolation, the Schöder number, and the N-kings problem, SIAM J. Discrete Math. 2 (1991), 275–280.

[13] V. R. Pratt: Computing permutations with double-ended queues, parallel stacks and parallel queues, Proc. ACM Symp. Theory of Computing 5 (1973), 268–277.

[14] R. Simion, F. W. Schmidt: Restricted permutations, Europ. J. Combinatorics 6 (1985), 383–406.

[15] D. A. Spielman and M. Bóna: An Infinite Antichain of Permutations, Note N2, Elec. J. Comb. 7(1), 2000.

[16] Z. E. Stankova: Forbidden subsequences, Discrete Math. 132 (1994), 291–316.

[17] Z. E. Stankova: Classification of forbidden subsequences of length 4, European J. Combin. 17 (1996), no. 5, 501–517.

[18] R. E. Tarjan: Sorting using networks of queues and stacks, Journal of the ACM 19 (1972), 341–346.

[19] J. West: Generating trees and forbidden sequences, Discrete Math. 157 (1996), 363–374.