On permutation pattern classes with two restrictions only

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Abstract

Permutation pattern classes that are defined by avoiding two permutations only and which contain only finitely many simple permutations are characterized and their growth rates are determined.

1 Introduction

Permutation pattern classes are sets of permutations that are closed under taking subpermutations. A permutation $\pi = p_1 p_2 \cdots p_m$ is a *subpermutation* of (or is *involved* in) a permutation $\sigma = s_1 s_2 \cdots s_n$ if σ has a subsequence $s_{i_1} s_{i_2} \cdots s_{i_m}$ that is order isomorphic to π (that is, $p_1 p_2 \cdots p_m$ and $s_{i_1} s_{i_2} \cdots s_{i_m}$ are in the same relative order). For example 312 is a subpermutation of 41532 because of the subsequence 413. Notice that 123 is not a subpermutation of 41532 and we say that 41532 *avoids* 123.

Permutation pattern classes can equivalently be defined as sets of permutations which avoid certain forbidden subpermutations. In this case we normally use the smallest possible forbidden set, sometimes called the *basis*. The basis is just the set of permutations that are minimal (under the subpermutation order) that do not belong to the class; it may or may not be finite. We use the notation $\mathrm{Av}(\alpha, \beta, \ldots)$ to denote the pattern class of all permutations that avoid each of $\alpha, \beta, \ldots$.

There are 8 symmetries on the set of pattern classes that preserve their properties. They are generated by reversal ($s_1 s_2 \cdots s_n \longrightarrow s_n \cdots s_2 s_1$), complementation ($s_1 s_2 \cdots s_n \longrightarrow n+1-s_1 \cdots n+1-s_n$), and inversion. For any of these symmetries ϕ we have $\mathrm{Av}(\alpha, \beta, \ldots)^\phi = \mathrm{Av}(\alpha^\phi, \beta^\phi, \ldots)$ and this often gives a reduction in the number of cases one has to consider.

The usual problem that one wishes to solve of a pattern class is to enumerate it: to find the number a_n of permutations of each length n that it contains. For nearly all pattern classes this problem is too hard to be solved exactly and so less precise information is sought. For example we might try to find properties of the generating function $\sum a_n x^n$ or asymptotic information about a_n. By the main result of [8] we know that (unless the pattern class contains all permutations)

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = K$$

is finite and so K^n is an approximate estimate for a_n. For this reason K is called the *growth rate* of the class. For some pattern classes (including those with a single basis element [4]) the lim sup is known to be a true limit; it seems likely that this is true in general.

In this paper we shall show how to find the growth rate of a 3-parameter family of pattern classes. We shall approach the problem using simple permutations proving another theorem of independent interest. To state it we first recall the basic facts about simple permutations and their importance for permutation pattern classes.

An *interval* of a permutation $\beta = b_1 \cdots b_n$ is a subsequence of consecutive positions $b_i b_{i+1} \cdots b_j$ such that the set $\{b_i, b_{i+1}, \ldots, b_j\}$ is a set of consecutive values. Every permutation has singleton intervals and the entire permutation is also an interval but, if these are the only intervals, the permutation is said to be *simple*. For example 246135 is simple but 5174236 is not simple since it has an interval 423.

If a permutation α is a concatenation $\bar{\tau}_1\bar{\tau}_2\cdots\bar{\tau}_r$ where each $\bar{\tau}_i$ is an interval order isomorphic to a permutation τ_i then, for any two intervals $\bar{\tau}_i, \bar{\tau}_j$, either all the terms of $\bar{\tau}_i$ are less than all the terms of $\bar{\tau}_j$ or they are all greater; we express this as $\tau_i < \tau_i$ or $\tau_i > \tau_j$ respectively. Therefore we can use the notation

$$\alpha = \sigma[\tau_1, \ldots, \tau_r]$$

where σ is the permutation defined by the relative order of the intervals $\bar{\tau}_i$. It is proved in [1] that every permutation has a representation of this form in which σ is a simple permutation. Indeed, so long as $r > 2$ (and σ is simple), the permutations τ_i are also uniquely determined.

Pattern classes that contain only a finite number of simple permutations are particularly tractable [1] in that they have a finite basis, are partially well-ordered, and have algebraic generating functions. Thus it is of great interest to know when a pattern class has just a finite number of simple permutations. A decision procedure to decide this question (given a finitely based class) has recently been given [6]; in principle it gives a bound on the length of the longest simple permutation in the class (when this exists) but the bound is almost certainly very excessive.

Pattern classes of the form Av(α) have a finite number of simple permutations only if $\alpha = 1, 12, 231$ (or a permutation equivalent to one of these by a symmetry). Our first result is about pattern classes defined by two avoided permutations. To state it we adopt the following notation. The increasing permutations $1\ 2 \cdots k-1\ k$ and decreasing permutations $k\ k-1 \cdots 2\ 1$ will be denoted by ι_k and δ_k respectively. Furthermore, if λ, μ are any permutations then $\lambda \oplus \mu$ (respectively $\lambda \ominus \mu$) will denote the permutation $\tilde{\lambda}\tilde{\mu}$ where the segments $\tilde{\lambda}, \tilde{\mu}$ are isomorphic to λ, μ and every term of $\tilde{\lambda}$ is less than (respectively greater than) every term of $\tilde{\mu}$.

Theorem 1 *$Av(\alpha, \beta)$ has a finite number of simple permutations if and only if the pair α, β is equivalent by a symmetry and possible exchange of α with β to one of the following types:*

1. $\alpha = 231$
2. $\alpha = 2413, \beta = 3142$
3. $\alpha = \delta_k, \beta = \iota_a \oplus \theta \oplus \iota_b$ *where θ is one of the following permutations:* $1, 21, 231$.

It is then immediate by the results of [1] that the pattern classes listed in the three cases of the theorem have algebraic generating functions. In fact, for the first type, the generating function is actually a rational function $p(x)/q(x)$ and an efficient recursive procedure for calculating it is given in [9]; the rate of growth of the class is, of course, then determined as the reciprocal of the smallest root of the polynomial $q(x)$. The simple permutations of these classes are $1, 12, 21$.

The second type is the class of all separable permutations (see [5]) and its generating function is

$$\frac{1 - x - \sqrt{1 - 6x + x^2}}{2}$$

with growth rate $3+2\sqrt{2}$. The simple permutations of this class are also $1, 12, 21$. We shall come across this class later and so we recall the definition of separable permutations as those permutations that can be obtained from the permutation 1 by repeated use of the operations $\oplus, \ominus$. For example, 31254 is separable since each of $12 = 1 \oplus 1, 312 = 1 \ominus 12, 21 = 1 \ominus 1$ are separable and $31254 = 312 \oplus 21$. Another way of defining them (see [3]) is as those permutations obtained from the permutation 1 by repeatedly replacing symbols i by $i\ i+1$ or by $i+1\ i$ (with appropriate relabelling of the other symbols). For example we obtain 31254 in this way by the operations

$$\underline{1} \longrightarrow \underline{1}2 \longrightarrow 2\underline{1}3 \longrightarrow 312\underline{4} \longrightarrow 31254$$

(the symbol being replaced at each step is underlined).

For the third of the types in Theorem 1 we note that if $\theta = 1$ then $\iota_a \oplus 1 \oplus \iota_b = \iota_{a+b+1}$ and $\text{Av}(\delta_k, \iota_{a+b+1})$ is finite by the Erdős-Szekeres result [7]; since we use this result several times later on we state it explicitly:

Proposition 1 *$Av(\delta_k, \iota_\ell)$ is finite and its longest permutations have length $(k-1)(\ell-1)$.*

In the case $\theta = 21$ we can appeal to one of the results of [2]: the number of permutations of degree n in $\mathrm{Av}(\delta_k, \iota_a \oplus 21 \oplus \iota_b)$ is a polynomial in n; so here the growth rate is 1. The remaining case $\theta = 231$ is the subject of our second result.

Theorem 2 *The growth rate of $Av(\delta_k, \iota_a \oplus 231 \oplus \iota_b)$ is independent of a, b and hence is equal to the growth rate of $Av(\delta_k, 231)$.*

The growth rate of $\mathrm{Av}(\delta_k, 231)$ can be determined by the methods of [9]. For completeness we give the argument which, in this special case, is quite easy. A permutation of length $n > 0$ in $\mathrm{Av}(\delta_k, 231)$ can be written as $\lambda n \mu$ where every term of λ is less than every term of μ. Clearly, $\mu \in \mathrm{Av}(\delta_{k-1}, 231)$ (and, of course, $\lambda \in \mathrm{Av}(\delta_k, 231)$). Hence, if $f_k = f_k(x)$ denotes the generating function of $\mathrm{Av}(\delta_k, 231)$, we have $f_1 = 1$ and, for $k > 1$,

$$f_k = 1 + x f_k f_{k-1}$$

which gives

$$f_k = \frac{1}{1 - x f_{k-1}}$$

We then find that f_k is a rational function q_{k-1}/q_k of x where $q_1 = 1, q_2 = 1 - x$ and, for $k > 2$,

$$q_k = q_{k-1} - x q_{k-2}$$

An easy inductive argument then proves that

$$q_k = \sum_i \binom{k-i}{i} (-x)^i$$

The first few of these polynomials are $q_3 = 1 - 2x, q_4 = 1 - 3x + x^2, q_5 = 1 - 4x + 3x^2$ and their smallest zeros ($x = 1/2, x = (3 - \sqrt{5})/2, x = 1/3$) give growth rates $2, (3 + \sqrt{5})/2, 3$.

The argument in the proof of Theorem 1 gives the rough upper bound

$$7^{k-1}(a+b)k/2$$

on the length of the simple permutations in $\mathrm{Av}(\delta_k, \iota_a \oplus 231 \oplus \iota_b)$.

2 Proof of Theorem 1

We begin by giving some infinite families of simple permutations. They are presented as plots of points in the (x, y)-plane: $p_1 p_2 \cdots p_n$ is represented as the

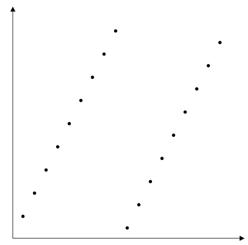

Figure 1: The family F_1 of permutations $246 \cdots 2n\ 135 \cdots 2n-1$. They all avoid $321, 2143$ and 3142. The family F_1^{-1} of inverse permutations all avoid $321, 2143$ and 2413

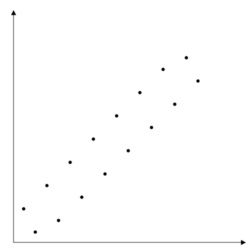

Figure 2: The family F_2 of permutations $31527496 \cdots 2n-1\ 2n-4\ 2n\ 2n-2$. They all avoid $321, 4123, 2341$ and 3412.

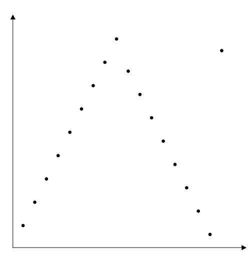

Figure 3: The family F_3 of permutations $246 \cdots 2n\ 2n-3\ 2n-5 \cdots 1\ 2n-1$. They all avoid $3412, 2143, 3142, 2134$ and 4132.

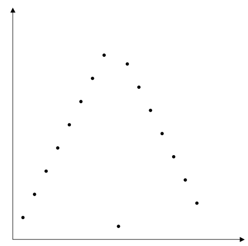

Figure 4: The family F_4 of permutations $246 \cdots 2n\ 1\ 2n-1\ 2n-3 \cdots 53$. They all avoid $2134, 4213, 1324, 3124, 4312, 4231, 3241$ and 1423

set of n points (i, p_i). These diagrams (Figure 1, Figure 2, Figure 3, and Figure 4) make it easy to check (as also verified in [6]) that the permutations are indeed simple and that they avoid the various permutations of small length stated.

Lemma 2 *If Av$(321, \alpha)$ has only finitely many simple permutations then α has one of the following forms:*

1. $\alpha = \iota_a$,
2. $\alpha = \iota_a \oplus 21 \oplus \iota_b$,
3. $\alpha = \iota_a \oplus 231 \oplus \iota_b$,
4. $\alpha = \iota_a \oplus 312 \oplus \iota_b$.

Proof: If either of 2143 or 3142 is a subpermutation of α then $\mathrm{Av}(321, \alpha)$ would contain (respectively) $\mathrm{Av}(321, 2143)$ or $\mathrm{Av}(321, 3142)$ and so would contain the infinite family F_1 of simple permutations which is impossible. Similarly, if 2413 was a subpermutation of α, $\mathrm{Av}(321, \alpha)$ would contain $\mathrm{Av}(321, 2413)$ which contains the infinite family F_1^{-1}.

Again similarly, none of $4123, 2341, 3412$ are subpermutations of α since, if they were, then $\mathrm{Av}(321, \alpha)$ would contain the infinite family F_2.

However, once we know that α avoids $\{2143, 3142, 2413, 4123, 2341, 3412\}$ it is virtually trivial to prove that α has one of the forms claimed. ∎

This allows us to prove the more general

Lemma 3 *If Av(α, β) has finitely many simple permutations then the pair α, β is equivalent under symmetry and exchange of α with β to one of those listed in Theorem 1.*

Proof: If either α or β has length 3 then we can appeal to Lemma 2 (if one is monotonic) or (otherwise) obtain the first case of Theorem 1.

Suppose next that neither is monotonic and both have length at least 4. If neither have a monotone subsequence of length 3 then, necessarily, they have length exactly 4 (Proposition 1) and are easily seen to lie in $\{3142, 2413, 3412, 2143\}$. Of the 6 possible pairs in this set we need only consider the pairs $\{3142, 2413\}$, $\{3412, 2143\}$, $\{3142, 3412\}$ since the 3 other pairs are equivalent to one of these by symmetry. However, each of $\mathrm{Av}(3412, 2143)$ and $\mathrm{Av}(3142, 3412)$ contain the simple family F_3 and hence only $\mathrm{Av}(3142, 2413)$ is possible which is the second case of Theorem 1.

Next we assume that one of α or β (α say) has a decreasing subsequence of length 3 but that neither is monotonic. Then

$$\mathrm{Av}(321, \beta) \subseteq \mathrm{Av}(\alpha, \beta)$$

and so β is of one of the types in Lemma 2. However, in all of those cases, β has a subsequence isomorphic to 123. Hence

$$\mathrm{Av}(\alpha, 123) \subseteq \mathrm{Av}(\alpha, \beta)$$

and then Lemma 2 tells us that α^R (the reverse of α) is also of one of the types of Lemma 2.

We select subpermutations α_0, β_0 of α, β of length 4, neither monotonic. Then we have

$$\mathrm{Av}(\alpha_0, \beta_0) \subseteq \mathrm{Av}(\alpha, \beta)$$

where α_0^R, β_0 are each of one of the types of Lemma 2. The possibilities for α_0 are therefore

$$4312, 4231, 3421, 2431, 3241, 4132, 4213$$

and for β_0 are

$$2134, 1324, 1243, 1342, 1423, 2314, 3124$$

If we can show that every possible $\mathrm{Av}(\alpha_0, \beta_0)$ has an infinite number of simple permutations we shall have excluded this case. However, this is relatively easy since not all the 49 combinations (α_0, β_0) need to be examined. To within symmetry and exchange of α_0 and β_0 we have only the following cases:

$$\begin{array}{lll} \mathrm{Av}(4312, 2134), & \mathrm{Av}(4312, 1324), & \mathrm{Av}(2134, 4213), \\ \mathrm{Av}(1324, 4213), & \mathrm{Av}(3124, 4213), & \mathrm{Av}(3241, 1423), \\ \mathrm{Av}(4213, 1423), & \mathrm{Av}(4231, 1324), & \mathrm{Av}(4132, 2134). \end{array}$$

The first 8 of these classes all contain the infinite family F_4 and the last contains F_3.

There remains only the situation where one of α, β is monotonic, say α is decreasing. Then 321 is a subpermutation of α so that $\mathrm{Av}(321, \beta)$ is contained in $\mathrm{Av}(\alpha, \beta)$ and Lemma 2 proves that we obtain the third case of Theorem 1. ■

Lemma 3 proves one half of Theorem 1. We now have to establish that the classes listed in the theorem do indeed have a finite number of simple permutations. For the classes $\mathrm{Av}(231, \beta)$ and $\mathrm{Av}(2413, 3142)$ we have already remarked that the only simple permutations in these classes are $1, 12, 21$. We have also already discussed the cases $\text{Av}(\delta_k, \iota_a \oplus 1 \oplus \iota_b)$ and $\text{Av}(\delta_k, \iota_a \oplus 21 \oplus \iota_b)$ and it remains to handle the case $\text{Av}(\delta_k, \iota_a \oplus 231 \oplus \iota_b)$. For convenience we write $\alpha_{ab} = \iota_a \oplus 231 \oplus \iota_b$.

We shall prove that $\text{Av}(\delta_k, \alpha_{ab})$ has only a finite number of simple permutations by giving an upper bound on the length of its simple permutations in terms of k, a, b. To do this we shall consider an arbitrary simple permutation $\theta \in \text{Av}(\delta_k, \alpha_{ab})$. Since θ avoids δ_k it has a decomposition into $k-1$ increasing subsequences (which we shall call *chains*). For convenience we shall use the (left) greedy decomposition. In this decomposition the first chain is the sequence of the left to right maxima (a term is a left to right maximum if it has no larger predecessors). Since the sequence of remaining terms avoids δ_{k-1} the second and subsequent chains can be defined iteratively; thus the ith chain is the sequence of left to right maxima of those terms not among the first $i-1$ chains. We set θ_i to be the sequence of terms in the first i chains and put $u_i = |\theta_i|$. Then $u_{k-1} = |\theta|$ and, by convention, we let $u_0 = 0$.

Lemma 4 *For $0 \leq i \leq k-2$ we have $u_{i+1} \leq 7u_i + a + b + 3(a+b+4)(k-i-2) + 1$ (with u_0 defined as 0).*

Proof: Consider Figure 5. This represents the subpermutation of θ defined by the terms of $\theta - \theta_i$. It has $t = a + b + p$ left to right maxima (represented by the small black circles) which comprise the $(i+1)$th chain of θ. The initial a and final b of these are shown (with $a = 3, b = 4$); we shall bound the number p of intermediate ones. The figure also defines various regions marked as V_j (vertical strips), H_j (horizontal strips), S_j and T. The diagram does not show the u_i terms in the first i chains of θ but these would all lie above the zigzag line.

The region T is empty from the α_{ab} avoidance.

The points in $\theta - \theta_{i+1}$ (i.e. the points in the regions of Figure 5 not among the left to right maxima shown there) form a subsequence that avoids δ_{k-i-1} and its terms therefore fall into $k-i-2$ chains. In each of these chains we call the initial a and final b points the *exterior* points of the chain, the others being called the *interior* points.

Let us first suppose that there are more than $2(k-i-2)$ V-regions that contain interior points. Then, by the pigeon-hole principle, there will be at least 3 V-regions that contain interior points of *the same* chain. These points are shown as x, y, z in Figure 6 and u, v, w are the immediately preceding left to right maxima of $\theta - \theta_i$. Now the initial a exterior points of the chain that contains x, y, z, together with points v, w, z, and the final b left to right maxima of Figure 6 form a subsequence order isomorphic to α_{ab}. This is a contradiction.

Hence at most $2(k-i-2)$ V-regions contain interior points. A similar argument proves also that at most $2(k-i-2)$ H-regions contain interior points and so there are at most $4(k-i-2)$ V-regions and H-regions in all that contain interior points. Also, since there are at most $(a+b)(k-i-2)$ exterior points at most $(a+b)(k-i-2)$ V-regions and H-regions in all contain exterior points. Hence

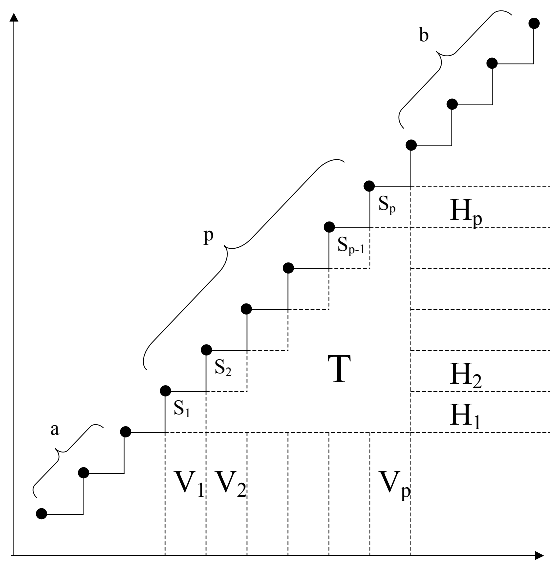

Figure 5: The subpermutation $\theta - \theta_i$ and its left to right maxima

there are at most $(a+b+4)(k-i-2)$ non-empty V-regions and H-regions in all.

Suppose some S_j is non-empty. Then, in order that the region S_j together with its immediately preceding left to right maximum not be an interval, there must be some term within θ whose value or position prevents this. Such a term must either lie in θ_i (i.e. among the u_i points of the first i chains), or lie in one of V_j or H_j. At most $2u_i$ of the non-empty S_j are prevented from being intervals by the points in θ_i (since a point of θ_i can split at most two of the S_j, one by value and one by position). The remainder are prevented from being empty by non-empty V or H regions of which there are at most $(a+b+4)(k-i-2)$. Hence there cannot be more than

$$N = 2u_i + (a+b+4)(k-i-2)$$

non-empty S-regions.

The number of empty S-regions is therefore at least $p - N$. Of these empty S-regions at most $N+1$ of them are followed by a non-empty S-region or by a point following S_p (should S_p happen to be empty) and so there are at least $p - 2N - 1$ pairs of consecutive empty S-regions.

Consider some pair S_j, S_{j+1} of consecutive empty S-regions with immediately preceding left to right maxima m_j, m_{j+1}. In order that $m_j m_{j+1}$ is not an interval at least one of V_j and H_{j+1} must be non-empty or it must be separated by one of the u_i points in the first i chains (again such points can separate at

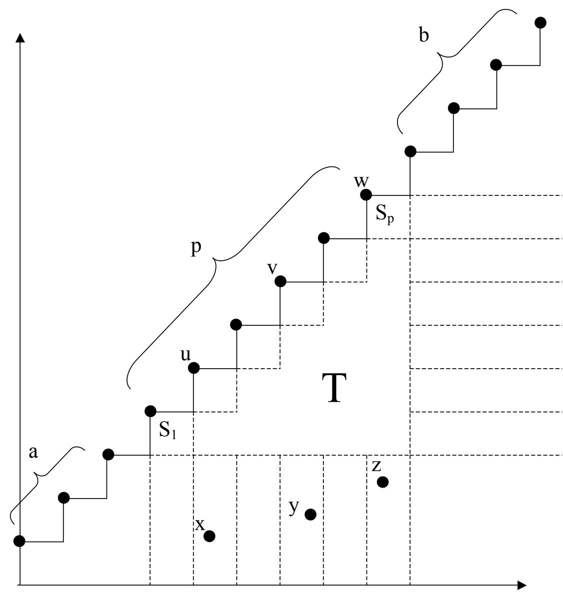

Figure 6: 3 V-regions containing 3 interior points

most two such pairs). Hence, if $p - 2N - 2u_i > (a+b+4)(k-i-2)$, we would have more than $(a+b+4)(k-i-2)$ non-empty V-regions and H-regions. This contradiction proves that

$$p - 2N - 1 - 2u_i \leq (a+b+4)(k-i-2)$$

leading to

$$p \leq 3(a+b+4)(k-i-2) + 6u_i + 1$$

and, as $u_{i+1} = u_i + a + b + p$ this completes the proof of the lemma. ∎

We can now complete the proof of Theorem 1. The recurrence in the previous lemma tells us that

$$|\theta| = |\theta_{k-1}| \leq 7^{k-1}(a+b)k/2$$

and hence there are only finitely many simple permutations in $\mathrm{Av}(\delta_k, \alpha_{ab})$.

Finally, we note that by using the decision procedure in [6] the proof of Theorem 1 could be shortened. However, our approach gives a rather better bound on the lengths of the simple permutations in question; it remains open whether this bound is optimal or close to optimal.

3 Proof of Theorem 2

We continue with the notation of the previous section and begin by proving a slightly stronger result about $\mathrm{Av}(\delta_k, \alpha_{ab})$. To state it we require one more definition. We say that a permutation σ is *strongly irreducible* if it has no segments of the forms $i\ i+1$ or $i+1\ i$ (in other words, it has no intervals of size 2).

Lemma 5 *$Av(\delta_k, \alpha_{ab})$ has only a finite number of strongly irreducible permutations.*

Proof:

We prove the result by induction on k it being obviously true for $k \leq 2$. Strongly irreducible permutations in $\mathrm{Av}(\delta_k, \alpha_{ab})$ of length greater than 1 have one of three forms (see [1]):

1. $\theta \ominus \phi$. Here θ and ϕ are themselves strongly irreducible and lie in $\mathrm{Av}(\delta_{k-1}, \alpha_{ab})$. So, by the inductive hypothesis, there are only finitely many possibilities for θ and ϕ.
2. $\sigma[\tau_1, \tau_2, \ldots, \tau_r]$ where σ is a simple permutation of length $r \geq 4$. Since σ is simple, for each τ_i, there is either some preceding τ_j with $\tau_j > \tau_i$ or some succeeding τ_j with $\tau_i < \tau_j$; so again each $\tau_i \in \mathrm{Av}(\delta_{k-1}, \alpha_{ab})$. By the inductive hypothesis there are only finitely many possibilities for each τ_i and since $r = |\sigma|$ is bounded we have, again, only finitely many possibilities.
3. $\theta \oplus \phi$. In this case we write the permutation as

$$\theta_1 \oplus \theta_2 \oplus \cdots \oplus \theta_g$$

with g maximal (so that each θ_i cannot be further decomposed in this manner). We must have $g < a + b + 2$. To see this note that $\theta_{a+1} \oplus \theta_{a+2}$ either contains some subpermutation 231 or is separable. The latter would contradict the strong irreduciblity of the permutation whereas the former would imply a subpermutation α_{ab}. Since the θ_i cannot be further decomposed they must have one of the first two forms and so they must be bounded. Since g is bounded there are only finitely many permutations of this form.

■

From the point of view of growth rate this lemma allows us to replace the class $\mathrm{Av}(\delta_k, \alpha_{ab})$ by the more tractable class $\mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142)$.

Lemma 6 *$Av(\delta_k, \alpha_{ab})$ and its subclass $Av(\delta_k, \alpha_{ab}, 2413, 3142)$ have the same growth rate.*

Proof: Let F be the (finite) set of strong irreducibles in $\mathrm{Av}(\delta_k, \alpha_{ab})$. Then we have

$$\mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142) \subseteq \mathrm{Av}(\delta_k, \alpha_{ab}) \subseteq F \wr \mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142)$$

The class on the right hand side is a wreath product in the sense of [3] and, again by [3], its generating function is a polynomial in the generating function of $\mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142)$. Its growth rate is therefore the same as that of $\mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142)$ which completes the proof. ■

For convenience define

$$T_{kab} = \mathrm{Av}(\delta_k, \alpha_{ab}, 2413, 3142)$$

and

$$\tilde{T}_{kr} = \mathrm{Av}(\delta_k, \iota_r, 2413, 3142).$$

Furthermore, let t_{kab} and $\tilde{t}_{kr}$ denote the generating functions (in the variable x) of T_{kab} and $\tilde{T}_{kr}$.

Our aim is to show that the growth rate of T_{kab} is independent of a, b and thereby reduce the computation of the growth rate to the case $a = b = 0$. We shall do this by showing that its generating function (which is necessarily rational) has denominator whose roots depend only on k, i.e. the irreducible factors of the denominator are all factors of the denominator of t_{k00}.

Observe that every permutation in T_{kab} of length greater 1 has either the form $\lambda \oplus \mu$ or $\lambda \ominus \mu$ and these forms are mutually exclusive. The permutations that are not of the form $\lambda \oplus \mu$ (i.e. those of the form $\lambda \ominus \mu$ together with the permutation 1) will be called *plus indecomposable*; those not of the form $\lambda \ominus \mu$ will be called *minus indecomposable*. The subsets of T_{kab} of plus and minus indecomposable permutations will be denoted by T^+_{kab} and T^-_{kab} respectively with a similar notation for the plus and minus indecomposable permutations of $\tilde{T}_{kr}$. The ordinary generating functions of these sets will be denoted by $t_{kab}, t^+_{kab}, \tilde{t}_{kr}$ etc.

Assume that not both a, b are zero. We shall obtain recursive descriptions of the plus and minus indecomposable permutations from which we shall obtain equations for their generating functions. This follows the basic approach introduced in [1].

1. Plus indecomposables. Those of length greater than 1 have the (unique) form $\sigma \ominus \tau$ where σ is minus indecomposable. Because α_{ab} is minus indecomposable, such a permutation avoids α_{ab} if and only if each of σ and τ avoid α_{ab}. For some unique r with $1 \leq r \leq k$ we shall have σ involves δ_{r-1} but σ does not involve δ_r; and then τ must avoid δ_{k-r+1}. Then we shall have
$$\sigma \in T^-_{rab} \setminus T^-_{r-1,ab}, \text{ and } \tau \in T_{k-r+1,ab}$$

Notice that, as both σ, τ are non-empty, the values $r = 1, r = k$ do not arise. Furthermore, if $1 < r < k$ and the displayed conditions hold, then $\sigma \ominus \tau \in T^+_{kab}$.

2. Minus indecomposables. Those of length greater than 1 have the (unique) form $\sigma \oplus \tau$ where σ is plus indecomposable. Such a permutation avoids δ_k if and only if each of σ and τ avoid δ_k. There are now three disjoint cases:

   (a) σ avoids α_{a0} and avoids ι_a. Note that the second condition implies the first. Here, for some unique r with $1 < r \leq a$, σ involves ι_{r-1} and avoids ι_r (we cannot have $r = 1$ as σ is non-empty); and then τ must avoid $\alpha_{a-r+1,b}$. So

   $$\sigma \in \tilde{T}^+_{kr} \setminus \tilde{T}^+_{k,r-1} \text{ and } \tau \in T_{k,a-r+1,b}$$

   (b) σ avoids α_{a0} and involves ι_a. Here τ avoids α_{0b}. Thus

   $$\sigma \in T^+_{ka0} \setminus \tilde{T}^+_{ka} \text{ and } \tau \in T_{k0b}$$

   (c) σ involves α_{a0}. For some unique r, $1 \leq r < b$, σ involves $\alpha_{a,r-1}$ but avoids α_{ar}. Then τ avoids ι_{b-r+1} (and, as τ is non-empty, $r = b$ cannot arise). So

   $$\sigma \in T^+_{kar} \setminus T^+_{ka,r-1} \text{ and } \tau \in \tilde{T}_{k,b-r+1}$$

   If any of the displayed conditions hold then $\sigma \oplus \tau \in T^-_{kab}$.

Passing to generating functions:

$$\begin{aligned}
t_{kab} &= t^+_{kab} + t^-_{kab} - x \\
t^+_{kab} &= x + \sum_{r=2}^{k-1} (t^-_{rab} - t^-_{r-1,ab}) t_{k-r+1,ab} \\
t^-_{kab} &= x + \sum_{r=2}^{a} (\tilde{t}_{kr} - \tilde{t}_{k,r-1}) t_{k,a-r+1,b} + (t^+_{ka0} - \tilde{t}^+_{ka}) t_{k0b} \\
&\quad + \sum_{r=1}^{b-1} (t^+_{kar} - t^+_{ka,r-1}) \tilde{t}_{k,b-r+1}
\end{aligned}$$

The quantities $t_{kab}, t^+_{kab}, t^-_{kab}$ are all rational functions whilst $\tilde{t}_{kr}, \tilde{t}^+_{kr}$ are polynomials (since, by Proposition 1, they enumerate finite sets).

Call a generating function of the form $t_{kab}, t^+_{kab}, t^-_{kab}$ "good" if its denominator has only irreducible factors that divide the denominator of $\prod_{i=1}^{k} t_{i00}$. We can use the equations above to show, by induction on k, a, b, that t_{kab} is good for all a, b.

First we prove that $t_{ka0}, t^+_{ka0}, t^-_{ka0}$ are good. By definition this is true for $a = 0$. But, for $a > 0$, the second equation tells us that t^+_{ka0} is good; then the third equation tells us that t^-_{ka0} is good, and finally the first equation tells us that t_{ka0} is good.

It then follows by symmetry that t_{k0b} is good for all b.

But now the equations can be used in a similar inductive argument to prove that t_{kab} is good for all a, b.

The growth rate of T_{kab} is, say, $1/\rho$ where ρ is the smallest root of the denominator of t_{kab}. By what we have just proved ρ is a root of one of the denominators of some t_{i00} with $1 \leq i \leq k$. Let $1/\rho_1, 1/\rho_2, \dots, 1/\rho_k$ be the growth rates of $t_{100}, t_{200}, \dots, t_{k00}$; in other words each ρ_j is the smallest root of the denominator of t_{j00}. But, from the definition of T_{j00} as a pattern class, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$ and hence $\rho \geq \rho_k$. However, $T_{k00} \subseteq T_{kab}$. and so $1/\rho_k \leq 1/\rho$. This proves that $\rho = \rho_k$ and that

Lemma 7 *For all $a, b \geq 0$, $Av(\delta_k, \alpha_{ab}, 2413, 3142)$ and $Av(\delta_k, \alpha_{00}, 2413, 3142)$ have the same growth rate.*

Theorem 2 now follows from Lemmas 6 and 7.

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