Restricted permutations and the wreath product

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Abstract

Restricted permutations are those constrained by having to avoid subsequences ordered in various prescribed ways. A closed set is a set of permutations all satisfying a given basis set of restrictions. A wreath product construction is introduced and it is shown that this construction gives rise to a number of useful techniques for deciding the finite basis question and solving the enumeration problem. Several applications of these techniques are given.

Key words: restricted, forbidden, permutation, sequence, subsequence, basis, enumeration

1 Introduction

This article is a sequel to [1]. In that paper we studied the partial order "involvement" on permutations and argued that it should be studied through ideals called closed sets. Closed sets are a natural setting for many combinatorial and computational problems [2,3,7,11,12,16], particularly those concerned with avoided subsequences [6,13–15,17,18]. Two issues arise for closed sets: whether they have a finite basis, and the enumeration of the permutations of each length. Answers to these questions are generally arrived at by uncovering the combinatorial structure of the closed set. In this paper, as in [1], we are primarily concerned with developing tools to analyse this structure. These tools are illustrated by a number of applications to the finite basis and enumeration questions.

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We begin by recalling the basic definitions. If π and σ are permutations and π is order isomorphic to a subsequence of σ then we say that π is *involved* in σ and write $\pi \preceq \sigma$. For example $231 \preceq 13542$ because 231 is order isomorphic to the subsequence 352 in 13542. A set $\mathcal{X}$ of permutations is said to be *closed* if, whenever $\sigma \in \mathcal{X}$ and $\pi \preceq \sigma$, then $\pi \in \mathcal{X}$. Closed sets can be defined by "forbidden sets". More precisely, the *basis* $\mathcal{X}^\star$ of a closed set $\mathcal{X}$ is the set of permutations, minimal with respect to $\preceq$, that do not belong to it. Clearly, then $\mathcal{X}$ is exactly the set of permutations that do not involve any permutation of $\mathcal{X}^\star$.

Much research in this subject centres around enumeration problems. Let $\mathcal{X}_n$ denote the subset of $\mathcal{X}$ whose permutations have length n. One of the most significant open questions on closed sets was asked by Richard Stanley and Herbert Wilf: if $\mathcal{X}^\star$ is non-empty is there a constant k such that $\mathcal{X}_n$ has at most k^n permutations?

A more precise conjecture was made by Ira Gessel for closed sets with a finite basis: does $|\mathcal{X}_n|$ satisfy a recurrence equation with polynomial coefficients? Of course, one can make this conjecture for an arbitrary closed set but it would have the following seemingly unlikely consequence. It is known that there are uncountably many closed sets. On the other hand there are but countably many recurrence equations with polynomial coefficients. Therefore there would be an uncountable number of closed sets with the same numbers of permutations of each length.

A more general conjecture was made in [10] but, in [1], this was shown to be equivalent to Gessel's.

The decision problem for a closed set X is to decide of any permutation σ whether $\sigma \in X$. If X has a finite basis the problem is in the complexity class P although the set of deque sortable permutations provides a counterexample to the converse statement (see [9]). For this and other reasons it is an important problem to determine whether X is finitely based and to find its basis.

In section 2 we define basic terminology and in section 3 we develop a general construction on closed sets called the wreath product, and explore its most useful special cases. In section 4 we discuss how the construction respects the finite basis property while section 5 describes some enumeration formulae in terms of generating functions. In section 6 we give a number of applications of the general theory.

2 Terminology

In this section we introduce notation and definitions beginning with the following convention. When discussing a set X of permutations we shall freely use terminology such as $\sigma \in X$ when σ is *any* sequence of distinct integers. What we actually mean is that σ is order isomorphic to a permutation of X. Foundationally it would perhaps be more satisfactory to consider equivalence classes (under order isomorphism) of finite sequences of distinct integers for then every equivalence class would have a unique representative which was a permutation. However, the present terminology is now so well established that we have preferred not to do this.

We write permutations in "image" form as lists of integers sometimes separated by commas.

Any set of consecutive integers is called an *interval*.

If α and β are two sequences or sets and if $a < b$ for all $a \in \alpha$ and $b \in \beta$ then we write $\alpha < \beta$. The notation $\alpha > \beta$ is defined similarly. Next, we define a permutation to be *forward indecomposable* or simply *indecomposable* if it cannot be expressed as a concatenation $\alpha\beta$ with both α, β non-empty and with $\alpha < \beta$. This term was first defined by Avis and Newborn [4] who also defined a permutation to be *backward indecomposable* if it has no non-trivial expression as $\alpha\beta$ with $\alpha > \beta$. Furthermore we say that a permutation is *strongly indecomposable* if it is both forward and backward indecomposable.

The notion of indecomposability will play a key role in the paper. There is another notion, called irreducibility, that, in some sense, plays a dual role. A permutation which has no segment of the form $i, i+1$ will be called *irreducible*. Similarly, if it has neither a segment $i, i+1$ nor a segment $i+1, i$ it will be called *strongly irreducible*.

Associated with indecomposability and irreducibility are two concepts that apply to sets of permutations. A set $\mathcal{X}$ with the property that, whenever $\alpha, \beta \in \mathcal{X}$, with $\alpha < \beta$, then also the permutation $\alpha\beta \in \mathcal{X}$ is called *complete*. It is called *strongly complete* if $\alpha\beta \in \mathcal{X}$ in both the cases $\alpha < \beta$ and $\alpha > \beta$. Notice that this is our first use of the convention that $\alpha \in X$ may mean that α is order isomorphic to a permutation in X.

Also a set $\mathcal{X}$ of permutations is said to be *expanded* if whenever a permutation $\sigma = \alpha i \beta \in \mathcal{X}$ so is the permutation obtained by replacing i by $i, i+1$ and increasing all symbols greater than i in σ by 1. This permutation is called the (positive) expansion of σ at i. The set is called *strongly expanded* if it is invariant under these expansion operations and also those where i is replaced by $i+1, i$ (negative expansions).

The intersection of complete (respectively, strongly complete, expanded, strongly expanded) sets is again complete (respectively, strongly complete, expanded, strongly expanded). Therefore we may define the completion of a set $\mathcal{X}$ as the smallest complete set containing $\mathcal{X}$. Similarly we may define the strong completion, expansion, and strong expansion.

Clearly, the completion of $\mathcal{X}$ consists of all permutations of the form $\xi_1\xi_2\ldots\xi_k$ where each $\xi_i \in \mathcal{X}$ and $\xi_1 < \xi_2 < \ldots < \xi_k$. The most elementary example of this situation is the case that $\mathcal{X}$ is the trivial set $\mathcal{T}$ consisting of the single permutation 1. Obviously the completion of $\mathcal{T}$ is just the set $\mathcal{I}$ of all identity permutations.

To describe the strong completion we use binary trees. Each node of a binary tree will be labelled by a permutation and will be either a 'forward' node or a 'backward' node. The label on any internal forward (respectively backward) node N will be a permutation $\alpha\beta$ where $\alpha < \beta$ (respectively $\alpha > \beta$) and where α, β are the labels on the left and right children of N. We can view such a tree as a set of rules for constructing the root permutation from the leaf permutations. It is easy to see that the permutations σ in the strong completion of X are exactly those for which there exists a tree with σ at the root and with permutations in X at the leaves.

In the case of the trivial set $\mathcal{T}$ these trees will have all their leaves labelled with the permutation 1 but we could rename each of the leaf symbols by the corresponding symbol in the root permutation reading from left to right. In this case therefore the trees are essentially those which were introduced in [8] to define the set $\mathcal{S}$ of separable permutations. In other words $\mathcal{S}$ is the strong completion of $\mathcal{T}$.

The expansion and strong expansion of a set $\mathcal{X}$ are more easily described. The expansion of $\mathcal{X}$ is obtained by repeated positive expansions at each symbol in each permutation of $\mathcal{X}$ while for the strong expansion we must allow both positive and negative expansions. It is clear that the expansion of the trivial set $\mathcal{T}$ is again the set $\mathcal{I}$ and we also have

Lemma 1 *$\mathcal{S}$ is both the strong completion and the strong expansion of the trivial set $\mathcal{T}$.*

PROOF. Only the second characterisation of $\mathcal{S}$ needs proof. We temporarily let $\mathcal{S}'$ be the strong expansion of $\mathcal{T}$. Thus $\mathcal{S}'$ is that set of permutations which can be obtained from 1 by a series of positive and negative expansions.

If $\alpha, \beta \in \mathcal{S}'$ with $\alpha < \beta$ we can obtain $\alpha\beta$ starting from the expansion of 1 to $1, 2$, expanding 1 to α and expanding 2 to β. Thus $\alpha\beta \in \mathcal{S}'$ if $\alpha < \beta$. But also $\alpha\beta \in \mathcal{S}'$ if $\alpha > \beta$ by starting with the expansion of 1 to $2, 1$. This proves that

$\mathcal{S}'$ is complete and hence that $\mathcal{S} \subseteq \mathcal{S}'$.

On the other hand we can prove that $\mathcal{S}' \subseteq \mathcal{S}$ by showing that $\mathcal{S}$ is expanded and we do this by induction. Let σ be a typical element of $\mathcal{S}$ so that $\sigma = \alpha\beta$ with $\alpha, \beta \in \mathcal{S}$ and either $\alpha < \beta$ or $\alpha > \beta$. Any expansion of σ must occur either within α or within β. Assume the expansion occurs within β producing the permutation β' say. Then $\beta' \in \mathcal{S}$ by induction and so the expansion $\alpha\beta'$ of σ is therefore also in $\mathcal{S}$.

Finally, in this section, we recall the profile of a permutation as defined in [1]. Suppose that a permutation σ is expressed as $\sigma = \alpha_1\alpha_2 \dots \alpha_k$ where each α_i is a segment of increasing consecutive integers and k is minimal. Then we can choose symbols $a_i \in \alpha_i$ and consider the (necessarily irreducible) permutation α that is order isomorphic to $a_1 \dots a_k$. This depends on σ alone (not on the choice of the a_i) and is called the *profile* $r(\sigma)$ of σ. For example, the profile of 34512678 is 213.

3 The wreath product

This section introduces the main construction $A \wr B$ of the paper. By examining extremal cases of the construction we shall see why the ideas of indecomposability and irreducibility are necessary and why the sets $\mathcal{I}, \mathcal{S}$ are natural candidates to play the roles of A or B.

The *wreath product* of two (not necessarily closed) sets A, B of permutations is the set $A \wr B$ of all permutations $\sigma = \alpha_1\alpha_2 \dots \alpha_k$ where

(1) Each α_i is a rearrangement of an interval
(2) Each α_i is order isomorphic to a permutation of B (which, as previously noted, we sometimes abbreviate as $\alpha_i \in B$)
(3) If a_i is a symbol of α_i then $a_1a_2 \dots a_k$ is order isomorphic to a permutation of A (again we sometimes abbreviate this as $a_1a_2 \dots a_k \in A$). Notice that, because of the first condition, the order isomorphism class of $a_1a_2 \dots a_k$ is independent of the choice of each $a_i \in \alpha_i$.

The following two results are easy to verify.

Lemma 2 *If A and B are closed then $A \wr B$ is closed.*

Lemma 3 *The wreath product is associative: $(A \wr B) \wr C = A \wr (B \wr C)$.*

Since the representation of a permutation of $A \wr B$ may not be unique it is useful, in the case that both A and B are closed, to consider the two extremal values of k in the above definition.

I k maximal. Then, for all i, one of the following holds:

(a) α_i is strongly indecomposable.

(b) $\alpha_i = \beta\gamma$ with $\beta < \gamma$ but the positive expansion of $a_1 \ldots a_k$ at a_i is not in A.

(c) $\alpha_i = \beta\gamma$ with $\beta > \gamma$ but the negative expansion of $a_1 \ldots a_k$ at a_i is not in A.

The conditions simplify in some cases:

(i) If $A = \mathcal{I}$ the conditions are just the requirement that each α_i is indecomposable. Furthermore, every permutation of $\mathcal{I} \wr \mathcal{B}$ has a unique representation as $\alpha_1 \ldots \alpha_k$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_k$ and each α_i indecomposable.

(ii) If $A = \mathcal{S}$ the conditions are the requirement that each α_i is strongly indecomposable. Also, every permutation of $\mathcal{S} \wr \mathcal{B}$ has a unique representation as $\alpha_1 \ldots \alpha_k$ with $a_1 a_2 \ldots a_k \in \mathcal{S}$ and each α_i strongly indecomposable.

II k minimal. Then, for all i, one of the following holds:

(a) $a_i \neq a_{i+1} \pm 1$

(b) $a_i = a_{i+1} + 1$ but $\alpha_i \alpha_{i+1} \notin B$

(c) $a_i = a_{i+1} - 1$ but $\alpha_i \alpha_{i+1} \notin B$

Similarly, the conditions simplify in the cases $B = \mathcal{I}$ and $B = \mathcal{S}$.

(i) If $B = \mathcal{I}$ the conditions become that each $a_1 a_2 \ldots a_k$ should be irreducible. Moreover, every permutation of $\mathcal{A} \wr \mathcal{I}$ has a unique representation in the form $\alpha_1 \ldots \alpha_k$ where $a_1 a_2 \ldots a_k$ is irreducible and each α_i is an increasing sequence of consecutive integers.

(ii) If $B = \mathcal{S}$ the conditions become that each $a_1 a_2 \ldots a_k$ should be strongly irreducible. In this case, every permutation of $\mathcal{A} \wr \mathcal{S}$ has a unique representation in the form $\alpha_1 \ldots \alpha_k$ where $a_1 a_2 \ldots a_k$ is strongly irreducible and each $\alpha_i \in \mathcal{S}$.

It also follows from these considerations that

Lemma 4 *For any closed set $\mathcal{X}$ we have*

(1) $\mathcal{X} \wr \mathcal{I}$ is the expansion of $\mathcal{X}$
(2) $\mathcal{X} \wr \mathcal{S}$ is the strong expansion of $\mathcal{X}$
(3) $\mathcal{I} \wr \mathcal{X}$ is the completion of $\mathcal{X}$
(4) $\mathcal{S} \wr \mathcal{X}$ is the strong completion of $\mathcal{X}$

4 Bases

We have seen that $\mathcal{X} \wr \mathcal{I}$ and $\mathcal{I} \wr \mathcal{X}$ are particularly interesting special cases of the wreath product construction. We now investigate the question of whether they are finitely based.

Lemma 5 *(1) The closed set $\mathcal{X}$ is expanded (respectively, strongly expanded) if and only if every basis element is irreducible (respectively, strongly irreducible).*
(2) The closed set $\mathcal{X}$ is complete (respectively, strongly complete) if and only if every basis element is indecomposable (respectively, strongly complete).

PROOF.

(1) If $\mathcal{X}$ is expanded then a basis element cannot have a segment $i, i+1$ since then it would be a (positive) expansion of a permutation in $\mathcal{X}$ and so would belong to $\mathcal{X}$. Conversely, if none of the basis elements have any segments of the form $i, i+1$ expanding a permutation cannot introduce an involved basis permutation; thus expansions of permutations in $\mathcal{X}$ are also in $\mathcal{X}$.
(2) If $\mathcal{X}$ is complete then a basis element cannot have the form $\theta\phi$ with $\theta < \phi$ since then both $\theta, \phi \in \mathcal{X}$ and so $\theta\phi$ would belong to $\mathcal{X}$. Conversely, suppose that all the basis elements are indecomposable. Let $\alpha, \beta \in \mathcal{X}$ with $\alpha < \beta$. Then $\alpha\beta \in \mathcal{X}$ for otherwise there would be a basis element μ of $\mathcal{X}$ with $\mu \preceq \alpha\beta$; but, since μ is indecomposable, we would have $\mu \preceq \alpha$ or $\mu \preceq \beta$, a contradiction.

The variants where $\mathcal{X}$ is strongly expanded or strongly complete follow in the same way.

Lemma 6 *(1) π is a basis element of $A \wr \mathcal{I}$ if and only if π is minimal (under involvement) subject to*
(i) π is irreducible
(ii) $\pi \notin A$
(2) π is a basis element of $\mathcal{I} \wr A$ if and only if π is minimal (under involvement) subject to
(iii) π is indecomposable
(iv) $\pi \notin A$.

PROOF. For the first part we begin by noting that any permutation in $A \wr \mathcal{I}$ either belongs to A or is not irreducible. Now suppose π is a basis element of $A \wr \mathcal{I}$. Then, as $A \wr \mathcal{I}$ is expanded by Lemma 4, π is irreducible by Lemma 5. Also $\pi \notin A$ since $A \subseteq A \wr \mathcal{I}$ and $\pi \notin A \wr \mathcal{I}$. Thus π satisfies conditions (i) and (ii). Subject to this it is minimal since, if $\pi_1 \prec \pi$, then $\pi_1 \in A \wr \mathcal{I}$ and not both (i) and (ii) can hold for π_1.

Conversely, if π is not a basis element of $A \wr \mathcal{I}$ then either $\pi \in A \wr \mathcal{I}$ in which case not both (i) and (ii) can hold or π properly involves a basis element that, as above, does satisfy (i) and (ii) in which case π would not satisfy (i) and (ii) minimally.

The proof of the second part is almost the same. Here we observe instead that any permutation in $\mathcal{I} \wr A$ either belongs to A or is not indecomposable, and we use the fact that $\mathcal{I} \wr A$ is complete.

The obvious analogous statements hold for $A \wr \mathcal{S}$ and $\mathcal{S} \wr A$.

Lemma 7 *Let π be any permutation and σ a permutation minimal subject to*

(1) $\pi \preceq \sigma$
(2) σ is irreducible

Then $|\sigma| \leq 2|\pi| - 1$.

PROOF. Let $\pi = p_1 p_2 \dots p_k$ and $\sigma = s_1 s_2 \dots s_n$ where $s_{i_1} s_{i_2} \dots s_{i_k}$ is the subsequence of σ that is order isomorphic to π. We choose this to be the lexicographically left-most subsequence of σ that is order isomorphic to π.

We now construct another subsequence σ' of σ. This subsequence contains all the symbols $s_{i_1}, s_{i_2}, \dots, s_{i_k}$. In addition, each pair p_j, p_{j+1} causes 0 or 1 further symbols to be included in σ' as now described:

Suppose that p_j, p_{j+1} is not consecutive increasing. Such a pair requires no further symbol to be included in σ'. Notice that, irrespective of what other symbols we place into σ' during its construction, the symbols s_{i_j} and $s_{i_{j+1}}$ will always correspond to distinct symbols in the profile of σ'.

Now consider some p_j, p_{j+1} that is consecutive and increasing. If there was a symbol q positioned between s_{i_j} and $s_{i_{j+1}}$ such that $s_{i_j} < q < s_{i_{j+1}}$ we could match p_{j+1} to q rather than to $s_{i_{j+1}}$ to contradict the lexicographically left-most property of $s_{i_1} s_{i_2} \dots s_{i_k}$.

Therefore, if there is a symbol q positioned between s_{i_j} and $s_{i_{j+1}}$ it satisfies $q < s_{i_j}$ or $s_{i_{j+1}} < q$ and in either case we place q into σ' (but this is done for only one such q). This ensures that s_{i_j} and $s_{i_{j+1}}$ will correspond to distinct symbols in the profile of σ'.

If no such q exists then s_{i_j} and $s_{i_{j+1}}$ are positioned adjacently within σ. However, as σ is irreducible, s_{i_j} and $s_{i_{j+1}}$ cannot be consecutive in value. Therefore there must be, in σ, a symbol r with $s_{i_j} < r < s_{i_{j+1}}$. We place one such r into σ' and, again, this ensures that s_{i_j} and $s_{i_{j+1}}$ correspond to distinct distinct symbols in the profile of σ'.

We have constructed σ' so that its profile σ^* involves π. However, as $\sigma^* \preceq \sigma$ and σ was minimal, we must have $\sigma = \sigma^*$. Because of the construction, $|\sigma| \leq k + \ell$ where ℓ is the number of pairs p_j, p_{j+1} which are increasing and consecutive. Obviously, $\ell \leq k - 1$ and the proof is complete.

Theorem 8 *If $\mathcal{X}$ is a finitely based closed set then $\mathcal{X} \wr \mathcal{I}$ is also finitely based.*

PROOF. Let σ be a basis element of $\mathcal{X} \wr \mathcal{I}$. Then σ is irreducible (by Lemma 5 and because $\mathcal{X} \wr \mathcal{I}$ is fully expanded). As $\sigma \notin \mathcal{X}$ there is a basis element π of $\mathcal{X}$ with $\pi \preceq \sigma$. We now choose a permutation σ' minimal such that

(1) $\pi \preceq \sigma' \preceq \sigma$, and
(2) σ' is irreducible

But, by Lemma 7, $|\sigma'|$ is bounded in terms of $|\pi|$. However, Lemma 6, the properties of σ' guarantee that it is a basis element of $\mathcal{X} \wr \mathcal{I}$; therefore $\sigma = \sigma'$ and has bounded length.

On the other hand we have

Theorem 9 *Let $\mathcal{X}$ be the closed set with single basis element* 321654. *Then $\mathcal{I} \wr \mathcal{X}$ is not finitely based.*

PROOF. Consider the following set of permutations defined for $m > 2$:

$$\begin{aligned}\beta_m = & 3, 2, 5, 1, 7, 4, 9, 6, 11, 8, \ldots, 2i-1, 2i-4, 2i+1, 2i-2, \ldots \\ & 2m-1, 2m-4, 2m+2, 2m-2, 2m+1, 2m\end{aligned}$$

Apart from the first four and last four symbols the remainder of β_m is defined by interleaving odd-valued and even-valued symbols as indicated by the typical segment $2i-1, 2i-4, 2i+1, 2i-2$ given above.

By inspection we see that $3, 2, 1$ and $2m+2, 2m+1, 2m$ are the only decreasing subsequences of length 3 and therefore $3, 2, 1, 2m+2, 2m+1, 2m$ is the unique subsequence order isomorphic to 321654. In particular $\beta_m \notin \mathcal{X}$.

We also observe, since the segment 3251 overlaps 5174 which in turn overlaps 7496 etc., that β_m is indecomposable.

Now consider the effect of omitting any of the symbols of β_m. If we omit any of $3, 2, 1, 2m+2, 2m+1, 2m$ then 321654 will no longer be involved so the result will lie in $\mathcal{X}$. On the other hand if we omit one of the other symbols then the result will not be indecomposable since $3, 2, 1$ and $2m+2, 2m+1, 2m$ will be in different components. Hence both components will lie in $\mathcal{X}$ as they do not involve 321654 and the resulting permutation will therefore be in $\mathcal{I} \wr \mathcal{X}$. This proves that, subject to $\beta_m \notin \mathcal{X}$ and β_m indecomposable, β_m is minimal.

By Lemma 6 each β_m is a basis element of $\mathcal{I} \wr \mathcal{X}$ which is therefore not finitely based.

5 Enumeration

The principal tool for computing the sequence of numbers $|\mathcal{X}_n|$ for a set $\mathcal{X}$ is the ordinary generating function of the sequence; we call this simply the generating function of $\mathcal{X}$. Unless stated otherwise we consider these generating functions to have zero constant term. Thus the generating function for $\mathcal{I}$ is $x/(1-x)$ and the generating function for $\mathcal{S}$ is $s(x) = (1-x-\sqrt{1-6x+x^2})/2$ [8].

Theorem 10 *Let G, H be (not necessarily closed) sets of permutations and let $F = G \wr H$. Suppose that each permutation of F has a unique representation as a permutation in the wreath product. Let $f(x), g(x), h(x)$ be the ordinary generating functions of F, G, H. Then $f(x) = g(h(x))$.*

PROOF. Let $f(x) = \sum_{n=1}^{\infty} f_n x^n$, $g(x) = \sum_{n=1}^{\infty} g_n x^n$, and $h(x) = \sum_{n=1}^{\infty} h_n x^n$. Every permutation of F has a unique representation in the form $\phi = \gamma_1, \ldots, \gamma_k$ where each $\gamma_i \in H$ and, if $u_i \in \gamma_i$, we have $u_1 \ldots u_k \in G$. For a fixed $u_1 \ldots u_k$ and fixed lengths $n_i = |\gamma_i|$ there are $\prod_{i=1}^{k} h_{n_i}$ possibilities for ϕ and, letting the n_i vary over all positive integers that sum to n, we find

$$\sum_{n_1, n_2, \ldots, n_k} \prod h_{n_i}$$

such permutations (the summation being over all $n_1, \ldots, n_k$ that sum to n). But this is the coefficient of x^n in $(h_1 x + h_2 x^2 + \ldots)^k = h(x)^k$.

As k varies and $u_1 \ldots u_k$ varies over all the g_k permutations of length k we obtain all the permutations in F of length n so that f_n is the coefficient of x^n in $\sum g_k h(x)^k$. Therefore $f(x) = g(h(x))$ as required.

There are a number of corollaries of this theorem that follow in conjunction with the remarks preceding Lemma 4

Corollary 11 *Let F be a closed set, G its set of irreducible permutations, and let $f(x)$ and $g(x)$ be their generating functions. Then $f(x) = g(x/(1-x))$ is the generating function for $F \wr \mathcal{I}$.*

PROOF. $F \wr \mathcal{I} = G \wr \mathcal{I}$ and permutations of $G \wr \mathcal{I}$ have a unique representation.

Corollary 12 *Let F be a closed set, G its set of strongly irreducible permutations, and let $f(x)$ and $g(x)$ be their generating functions. Then $f(x) = g(s(x))$ is the generating function for $F \wr \mathcal{S}$.*

PROOF. $F \wr \mathcal{S} = G \wr \mathcal{S}$ and permutations of $G \wr \mathcal{S}$ have a unique representation.

Corollary 13 *Let F be a closed set, H its set of indecomposable permutations, and let $f(x)$ and $h(x)$ be their generating functions. Then $f(x) = h(x)/(1-h(x))$ is the generating function for $\mathcal{I} \wr F$.*

PROOF. $\mathcal{I} \wr F = \mathcal{I} \wr G$ and permutations of $\mathcal{I} \wr G$ have a unique representation.

Corollary 14 *Let F be a closed set, H its set of strongly indecomposable permutations, and let $f(x)$ and $h(x)$ be their generating functions. Then $f(x) = s(h(x))$ is the generating function for $\mathcal{S} \wr F$.*

PROOF. $\mathcal{S} \wr F = \mathcal{S} \wr G$ and permutations of $\mathcal{S} \wr G$ have a unique representation.

We conclude this section with a theorem that is useful in cases where we are able to enumerate permutations that are both irreducible and indecomposable. It requires the following lemma.

Lemma 15 *Let $\sigma = \alpha_1 \alpha_2 \ldots \alpha_k$ where each α_i is irreducible and indecomposable and $\alpha_i < \alpha_{i+1}$. Then σ is irreducible if and only if no two consecutive α_i have length 1.*

PROOF. The permutation σ will have a segment $j, j+1$ if and only if the symbol j is the final symbol of some α_i and the symbol $j+1$ is the initial symbol of α_{i+1}. But then j would be the largest symbol of α_i and $j+1$ would be the smallest symbol of α_{i+1}. Then, as α_i and α_{i+1} are indecomposable, they must each have length 1.

Theorem 16 *Let F be a complete closed set, K its set of irreducible indecomposable permutations, and let $f(x), k(x)$ be their generating functions. Then*

$$f(x) = \frac{(1-x)k(x/(1-x)) - x^2}{(1-x)k(x/(1-x)) - x^2 + x - 1}$$

PROOF. Let g_n be the number of irreducible permutations of length n in F and let q_n be the number of irreducible permutations of F that begin with 1. Then q_n enumerates permutations 1α where α is irreducible and does not begin with its lowest symbol 2. Since there are $g_{n-1} - q_{n-1}$ choices for α we have

$$g_{n-1} = q_{n-1} + q_n$$

The irreducible permutations of F are of two kinds. Those that begin with 1, of which there are q_n, and those that begin with an indecomposable segment of length $i \geq 2$, of which there are $k_i g_{n-i}$ if $2 \leq i < n$ and k_n if $i = n$. Hence

$$\begin{aligned} g_n &= q_n + \sum_{i=2}^{n-1} k_i g_{n-i} + k_n \\ &= q_n + \sum_{i=1}^{n-1} k_i g_{n-i} - g_{n-1} + k_n \\ &= \sum_{i=1}^{n-1} k_i g_{n-i} - q_{n-1} + k_n \end{aligned}$$

Adding this equation to the corresponding equation where n is replaced by $n-1$ gives

$$\begin{aligned} g_n + g_{n-1} &= \sum_{i=1}^{n-1} k_i g_{n-i} + \sum_{i=1}^{n-2} k_i g_{n-1-i} - q_{n-1} - q_{n-2} + k_n + k_{n-1} \\ g_n + g_{n-1} + g_{n-2} &= \sum_{i=1}^{n-1} k_i g_{n-i} + \sum_{i=1}^{n-2} k_i g_{n-1-i} + k_n + k_{n-1} \end{aligned}$$

We now multiply by x^n and sum from 2 to ∞ recalling that $g_0 = 0$, $g_1 = 1$, $k_0 = 0$, $k_1 = 1$. This gives

$$g(x) - x + xg(x) + x^2 g(x) + x^2 = k(x)g(x) + xk(x)g(x) + k(x) - x + xk(x)$$

Solving for $g(x)$ we find

$$g(x) = \frac{(1+x)k(x) - x^2}{1 + x + x^2 - (1+x)k(x)}$$

Finally, the proof is completed by using Corollary 11.

6 Applications

In this section we shall give a number of examples of the use of the wreath product and the associated ideas of indecomposability and irreducibility. We shall see that basis questions still need to be answered by rather ad hoc arguments but that enumeration questions can often exploit the generating function results very effectively.

6.1 Sorting with a stack of queues

We shall apply the theory in the case of the closed set $\mathcal{U}$ of stack sortable permutations whose basis is $\{231\}$ to the investigation of $\mathcal{U} \wr \mathcal{I}$. Note that $\mathcal{U} \wr \mathcal{I}$ has an interpretation in terms of data structures. It is the set of permutations that can be sorted by a stack in which the push operation can take any number of input symbols at a time, place them in a queue, and then place this queue on the stack. The pop operation always removes an entire queue and discharges its elements to the output in the natural way. Certainly, by Theorem 8, $\mathcal{U} \wr \mathcal{I}$ will be finitely based and, from Lemma 7, its basis elements have length at most 5. Then a case by case search (whose details we omit) shows that the basis is $\{2431, 3241, 2413, 3142\}$

Now let g_n be the number of irreducible stack sortable permutations. If σ is any one of these put $\sigma = \lambda n \mu$. Necessary and sufficient conditions on λ and μ are

(1) $\lambda < \mu$
(2) λ is an irreducible stack sortable permutation of length k, with $0 \le k \le n-1$ and if $k = n-1$ then λ does not end with its maximal symbol $n-1$
(3) μ is an irreducible stack sortable permutation of length $n-1-k$

In view of this we define p_n to be the number of irreducible stack sortable permutations of length n that end with n. Each one of these permutations has the form τn where τ is irreducible and does not end with $n-1$; hence there are $g_{n-1} - p_{n-1}$ possibilities for τ. Thus

$$g_{n-1} = p_{n-1} + p_n \tag{1}$$

The conditions on λ and μ give

$$\begin{aligned} g_n &= g_{n-1} + \sum_{k=1}^{n-2} g_k g_{n-1-k} + g_{n-1} - p_{n-1} \\ &= \sum_{k=1}^{n-2} g_k g_{n-1-k} + 2g_{n-1} - p_{n-1} \end{aligned}$$

Adding this equation to the corresponding equation in which n is replaced by $n-1$ and using Equation 1 we find

$$g_n + g_{n-1} = \sum_{k=1}^{n-2} g_k g_{n-1-k} + \sum_{k=1}^{n-3} g_k g_{n-2-k} + 2g_{n-1} + 2g_{n-2} - g_{n-2}$$

and therefore

$$g_n = \sum_{k=1}^{n-2} g_k g_{n-1-k} + \sum_{k=1}^{n-3} g_k g_{n-2-k} + g_{n-1} + g_{n-2}$$

Now we multiply throughout by x^n, sum from 2 to ∞, and rearrange terms to get

$$g(x)^2(x^2 + x) + g(x)(x^2 + x - 1) + x = 0$$

from which we obtain

$$g(x) = \frac{1 - x - x^2 - \sqrt{1 - 2x - 5x^2 - 2x^3 + x^4}}{2(x + x^2)}$$

Finally, using Corollary 11, we have

Theorem 17 *The generating function for $\mathcal{U} \wr \mathcal{I}$ is*

$$\frac{1}{2x}(1 - 3x + x^2 - \sqrt{1 - 6x + 7x^2 - 2x^3 + x^4}) =$$
$$x + 2x^2 + 6x^3 + 20x^4 + 70x^5 + 254x^6 + 948x^7 + 3618x^8 + \ldots$$

Continuing with the notation of the previous subsection we might now ask about $\mathcal{I} \wr \mathcal{U}$. However, because 231 is indecomposable, $\mathcal{U}$ is complete by Lemma 5. Therefore, by Lemma 4, $\mathcal{U} = \mathcal{I} \wr \mathcal{U}$. To get a more interesting example we consider instead the closed set $\mathcal{V}$ whose single basis element is 213 (the set of permutations that can be sorted into reversed order by a stack).

We begin by finding the basis elements of $\mathcal{I} \wr \mathcal{V}$ (despite the example in Theorem 9, $\mathcal{I} \wr \mathcal{V}$ is finitely based).

Theorem 18 *The basis of* $\mathcal{I} \wr \mathcal{V}$ *is* $\{4213, 2413, 3142, 3241\}$.

PROOF. Let π be a basis permutation. Then, by Lemma 6, π is indecomposable, $213 \preceq \pi$, and π is minimal with these properties. Suppose that π is not one of the permutations in the statement of the theorem; then none of them will be involved in π. We put

$$\pi = \alpha x \beta y \gamma z \delta$$

with xyz order isomorphic to 213. We choose z to be the earliest symbol after y that exceeds x, and x to be the latest symbol before y whose value is between y and z. Then we have

(i) If $g \in \gamma$ then $g < x$ (from the choice of z)
(ii) If $b \in \beta$ then $x > b$ or $b > z$ (since $x < b < z$ contradicts the choice of x)
(iii) If $b \in \beta$ then $b < z$ (otherwise $xbyz$ is order isomorphic to 2413)
(iv) If $b \in \beta$ then $x > b$ (from (ii) and (iii))
(v) If $a \in \alpha$ then $a < z$ (otherwise $axyz$ is order isomorphic to 4213)
(vi) If $d \in \delta$ then $y > d$ or $d > x$ (for, if $y < d < x$, then $xyzd$ is order isomorphic to 3142)
(vii) If $d \in \delta$ then $y < d$ (otherwise $xyzd$ is order isomorphic to 3241)
(viii) If $d \in \delta$ then $x < d$ (from (vi) and (vii))

These facts prove that $\beta y \gamma < x < z\delta$ and that $\alpha < z$. We now prove that $\alpha < \delta$. This will prove that $\alpha x \beta y \gamma < z\delta$ and hence that π is not indecomposable, a contradiction. So let $a \in \alpha$, $d \in \delta$ with $a > d$. Then, by (viii), $a > x$ and $axyd$ is order isomorphic to 4213 which is impossible.

To compute the generating function of $\mathcal{I} \wr \mathcal{V}$ we need the generating function $h(x) = h_1 x + h_2 x^2 + \ldots$ for the indecomposable permutations of $\mathcal{V}$. We can obtain this by noting that a permutation of $\mathcal{V}$, when expressed as

$\sigma = \alpha_1\alpha_2\ldots\alpha_k$, with $\alpha_1 < \alpha_2 < \ldots$ and all α_i indecomposable, must have $|\alpha_i| = 1$ for $i = 1, \ldots, k-1$ (this is because $213 \not\preceq \sigma$).

Hence, if $f(x) = \sum f_n x^n$ is the generating function for $\mathcal{V}$, we have $f_n = \sum_{k=1}^{n} h_k$ for $n > 0$. Thus $f(x) = \frac{h(x)}{1-x}$ and so $h(x) = f(x)(1-x)$. But it is known that $f(x) = \frac{1-2x-\sqrt{1-4x}}{2x}$ and so $h(x)$ can be calculated. Now, from Corollary 13, we obtain

Theorem 19 *The generating function of $\mathcal{I} \wr \mathcal{V}$ is given by*

$$\frac{-2x}{1-5x-\sqrt{1-4x}+\sqrt{1-4x}x+2x^2} =$$
$$x + 2x^2 + 6x^3 + 20x^4 + 69x^5 + 243x^6 + \ldots$$

6.3 Pop-stacks in series

In [4], Avis and Newborn introduced a data structure called a 'pop-stack'. Pop-stacks resemble ordinary stacks in having a push operation which transfers the next item of an input stream onto the top of the stack, but their pop operation empties the *entire* stack content into an output stream. They studied pop-stacks 'in series'. A series arrangement of m pop-stacks is one in which the (entire) set of items popped from the i^{th} stack is pushed onto the $(i+1)^{th}$. The items from the input stream are pushed one by one onto the first pop-stack, travel through the system of stacks, and the output stream is generated by the output of the last pop-stack. We say that a permutation is *m-feasible* if it can be sorted by such a system (a slight departure from the Avis-Newborn definition). A permutation is said to be *feasible* if it is m-feasible for some m. Avis and Newborn gave enumeration results for both m-feasible and feasible permutations. In this section we use our wreath product results to give a somewhat simpler treatment.

We have already defined the set $\mathcal{I}$ of identity permutations. There is a related set $\mathcal{R}$ of all reversed identity permutations $n, n-1, \ldots, 2, 1$. The sets $\mathcal{I}$ and $\mathcal{R}$ are easily seen to be the smallest infinite closed sets. We define the sets

$$A_m = \mathcal{I} \wr \mathcal{R} \wr \mathcal{I} \wr \ldots \ (m \text{ factors})$$

$$B_m = \mathcal{R} \wr \mathcal{I} \wr \mathcal{R} \wr \ldots \ (m \text{ factors})$$

Theorem 20 *(1) A_m is the set of $(m-1)$-feasible permutations*
(2) $\bigcup_{m=1}^{\infty} A_m$ is the set of feasible permutations and this is also the set $\mathcal{S}$

PROOF. To prove the first part of the theorem we prove, by induction, the stronger result that A_m is the set of permutations sortable in $m-1$ pop-stacks and B_m is the set of permutations reverse sortable in $m-1$ pop-stacks. This is clearly true for $m=1$. We let $m>1$ and make the inductive hypothesis that this result is true with m replaced by $m-1$.

Let $\sigma \in A_m = \mathcal{I} \wr B_{m-1}$. Then $\sigma = \alpha_1\alpha_2\ldots\alpha_r$ where each $\alpha_i \in B_{m-1}$ and $\alpha_1 < \alpha_2 < \ldots$ By the inductive hypothesis, each α_i can be reverse sorted by $m-2$ pop-stacks in series. Therefore we can transfer the items of α_1 onto an $(m-1)^{th}$ pop-stack sorted decreasingly from bottom to top and so output the items of α_1 in sorted order. Repeating this for $\alpha_2, \alpha_3, \ldots$ shows that σ can be sorted using $m-1$ pop-stacks in series. Similarly, if $\sigma \in B_m = \mathcal{R} \wr A_{m-1}$ then $\sigma = \alpha_1\alpha_2\ldots$, $\alpha \in A_{m-1}$ and $\alpha_1 > \alpha_2 > \ldots$; then we can imitate the proof above to demonstrate the reverse sortability of σ in $m-1$ pop-stacks.

Conversely, suppose that σ can be sorted by $m-1$ pop-stacks in series. The sequence of pops from the final pop-stack defines a segmentation $\beta_1\beta_2\ldots$ of the sorted output $1, 2, \ldots$. It is easily seen, from the defining serial property, that, if $u \in \beta_i$, $v \in \beta_j$ and $i < j$, then u must precede v in σ. Hence $\sigma = \alpha_1\alpha_2\ldots$ with $\alpha_1 < \alpha_2 < \ldots$, and each α_i can be reverse sorted onto the final pop-stack. By the inductive hypothesis $\alpha_i \in B_{m-1}$ so that $\sigma \in \mathcal{I} \wr B_{m-1} = A_m$. By a similar argument we can also show that a permutation that is reverse sortable by $m-1$ pop-stacks must belong to B_m.

For the second part it is clear from the first part that the set of of feasible permutations is $\bigcup_{m=1}^{\infty} A_m$. That this is also the set of separable permutations follows from the definition of $\mathcal{S}$ as the strong completion of $\mathcal{T}$.

Corollary 21 *The set of m-feasible permutations is finitely based*

PROOF. The proof of Theorem 8 can easily be adapted to prove that $X \wr \mathcal{R}$ is finitely based whenever X is finitely based and the corollary therefore follows by repeated application of Theorem 8.

Now let $a_m = a_m(x)$ be the generating function for the set A_m. Since the permutations of B_m are the reverses of those in A_m, a_m also enumerates the permutations of B_m. Furthermore, let c_m and d_m be the generating functions for the forward indecomposable and backward indecomposable permutations of A_m respectively; these are also the generating functions for the backward and forward indecomposable permutations of B_m.

Since the permutation of length 1 is the only permutation of A_m that is both forward and backward indecomposable and since every permutation of A_m is either forward or backward indecomposable we have $a_m + x = c_m + d_m$. Moreover, every permutation of A_m has the form $\alpha_1\alpha_2\ldots\alpha_r$ with each $\alpha_i \in$

B_{m-1} and $\alpha_1 < \alpha_2 \ldots < \alpha_r$. If such a permutation is forward indecomposable then $r = 1$ and it is a forward indecomposable permutation of B_{m-1}. Hence $c_m = d_{m-1}$. This gives $d_{m-1} + d_m = a_m + x$. By Corollary 13, $a_m = c_m/(1 - c_m)$ and these equations prove the following

Theorem 22 *Let $a_m = a_m(x)$ be the generating function of A_m. Then $a_m = d_{m-1} + d_m - x$ where d_m is defined by $d_1 = x/(1-x)$ and $d_m = x + d_{m-1}^2/(1 - d_{m-1})$ for $m > 1$.*

This theorem demonstrates that the generating functions $a_m(x)$ are all rational. Although they rapidly become rather complicated as m increases they can be calculated for small values of m quite readily. For example,

$$a_3(x) = \frac{x(1 - 2x + 2x^2)}{2x^3 - 4x^2 + 4x - 1}$$

from which the recurrence

$$f_n = 4f_{n-1} - 4f_{n-2} + 2f_{n-3}$$

as computed in [4] can be obtained from the denominator.

6.4 Pop-stacks in genuine series

The serial pop-stacks considered in the last section are not true serial structures since a pop from one of the pop-stacks entails the pushing of all popped items onto the next stack. In a genuinely serial construction we would have to save the output from one stack before subjecting it to the next pop-stack.

The set P of permutations that are sortable by a single pop-stack can be analysed using the results of the previous section. We have $\mathcal{P} = \mathcal{I} \wr \mathcal{R}$ and the generating function $a_2(x) = x/(1-2x)$; therefore there are 2^{n-1} pop-stack permutations of length n. Furthermore it is easily seen that $\{231, 312\}$ is the basis of $\mathcal{P}$.

This section considers the set $\mathcal{Q} = \mathcal{P}^2$, the closed set of permutations that can be sorted by two pop-stacks in genuine series. The permutations of length n in $\mathcal{Q}$ arise by multiplying two permutations in $\mathcal{P}$ together. A general context for such problems was discussed in [5].

Lemma 23 *$\mathcal{Q}$ is complete and expanded.*

PROOF. If $\sigma = \alpha\beta$ with $\alpha < \beta$ where both $\alpha, \beta \in \mathcal{Q}$ then σ can clearly be sorted by two pop-stacks in genuine series: α is first processed and sorted, followed by β. Thus $\sigma \in \mathcal{Q}$ and $\mathcal{Q}$ is complete.

Suppose next that $\sigma = \alpha i \beta \in \mathcal{Q}$ and consider the expansion $\alpha i, i+1 \beta$ of σ at i. We can sort this using two pop-stacks in genuine series by following the algorithm $\mathcal{A}$ for sorting σ but pushing both i and $i+1$ onto the first or second pop-stacks when $\mathcal{A}$ pushed i. Because the symbols i and $i+1$ remain together in this algorithm they emerge in the output as a segment $i, i+1$ and since $\mathcal{A}$ sorted σ the new algorithm sorts $\alpha i, i+1\beta$. Thus $\mathcal{Q}$ is expanded.

Lemma 24 *The number of irreducible indecomposable permutations of length n in $\mathcal{Q}$ is 1 if $n = 1, 2, 3$ and $2^{n-1} - 1$ for $n > 3$.*

PROOF. The permutations of $\mathcal{Q}$ arise by multiplying two pop-stack permutations together. They therefore arise by multiplying a permutation of the form $\alpha = \alpha_1 \dots \alpha_k$, where $\alpha_1 < \alpha_2 < \dots$ and each α_i is decreasing, by another pop-stack permutation. Multiplying α by a pop-stack permutation entails dividing α into disjoint segments $\beta_1, \beta_2 \dots$ and reversing each segment to obtain a permutation σ. If we require σ to be irreducible and indecomposable there are strong restrictions on $\beta_1, \beta_2, \dots$.

If σ is to be irreducible no β_j can contain a segment $r, r+1$. Therefore, apart from the initial and final symbol of each α_i, each symbol of an α_i must lie within a segment β_j of length 1. It remains to describe how the initial and final symbols of each α_i lie within the β_j.

Consider the final symbol of some α_i that is not the last symbol of σ itself. Since σ is to be indecomposable this symbol cannot be the last symbol of any β_j and so the segment β_j that contains it must also contain the first symbol of α_{i+1}. In particular, notice that this implies that any singleton α_i must lie in the same β_j as both the predecessor and successor symbol of α_i.

These observations prove that there is at most one irreducible indecomposable permutation that arises by multiplying α by another pop-stack permutation. They also show how to find the resulting permutation of $\mathcal{Q}$ when it exists. For example, if α is the permutation $321|4|5|76|8$, where the segments α_i are marked, then the segments β_j are as shown in $3|2|1457|68$ and the resulting permutation of $\mathcal{Q}$ is 32754186.

It remains to determine which permutations arising from this construction are actually irreducible and indecomposable, and how many are distinct. We leave to the reader the check that, if $n > 3$, the rule above produces only irreducible, indecomposable permutations; and to verify that all the permutations produced are distinct except for the ones generated from $\alpha = 12 \dots n$

and $\alpha = n \dots 21$ both of which produce $n \dots 21$. Since there are 2^{n-1} choices for α the lemma follows.

From the lemma we see that the generating function for the number of irreducible indecomposable permutations of $\mathcal{Q}$ is

$$k(x) = x - 2x^3 + x/(1-2x) - x/(1-x)$$

Now Theorem 16 can be applied and it gives

Theorem 25 *The generating function for the set Q is*

$$\begin{aligned} f(x) = & \frac{x - 7x^2 + 19x^3 - 21x^4 + 10x^5 - 6x^6}{1 - 9x + 31x^2 - 53x^3 + 44x^4 - 16x^5 + 6x^6} \\ = & x + 2x^2 + 6x^3 + 24x^4 + 102x^5 + 414x^6 + 1598x^7 + 5982x^8 + \dots \end{aligned}$$

Note 1 *We have also proved that Q is a finitely based closed set and found its basis explicitly. It consists of* 29 *permutations of lengths* 5 *and* 6*.*

6.5 Enumeration given the basis

As our final application we give a brief example of how one can sometimes go about solving the enumeration problem for a closed set that is given by its basis. In this example we use the idea of strong irreducibility.

Consider the set $\mathcal{W}$ whose basis is $\{3142, 24135, 52413, 13524\}$. As the result of a fairly short computation we find the set G of strongly irreducible permutations of $\mathcal{W}$. It turns out that, for each $n \geq 5$, there are exactly two such, and otherwise there is one of each length 1 and 4 but none of length 2 or 3. Thus the generating function for G is

$$g(x) = x + x^4 + 2x^5 + 2x^6 + \dots = x + x^4 + 2x^5/(1-x)$$

By Corollary 12 the generating function for $G \wr S$ is $g(s(x))$. But Lemmas 4 and 5 tell us that $G \wr S = \mathcal{W} \wr S = \mathcal{W}$. We deduce

Theorem 26 *The generating function for $\mathcal{W}$ is*

$$\frac{1 - 9x + 29x^2 - 30x^3 + 10x^4 - x^5 - \sqrt{1 - 6x + x^2}(1 - 6x + 13x^2 - 7x^3 + x^4)}{2x}$$

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