The Structure of (some!) Permutation Classes

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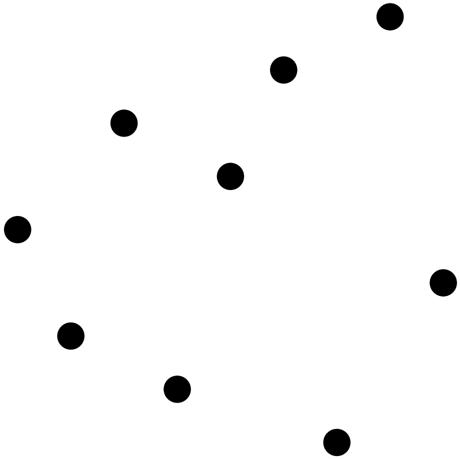
Permutation and Subpermutation



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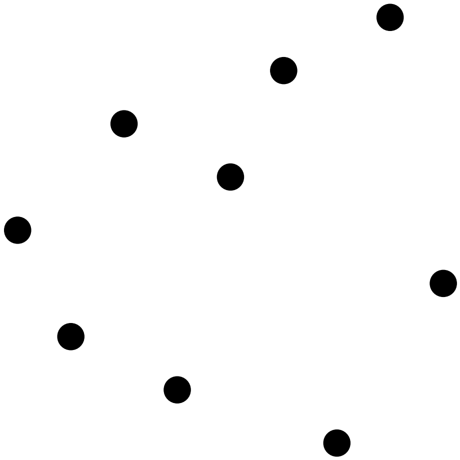

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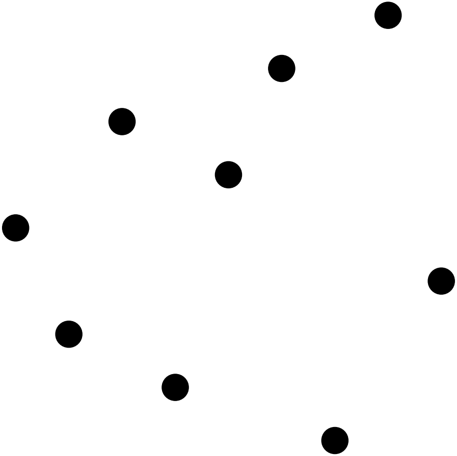

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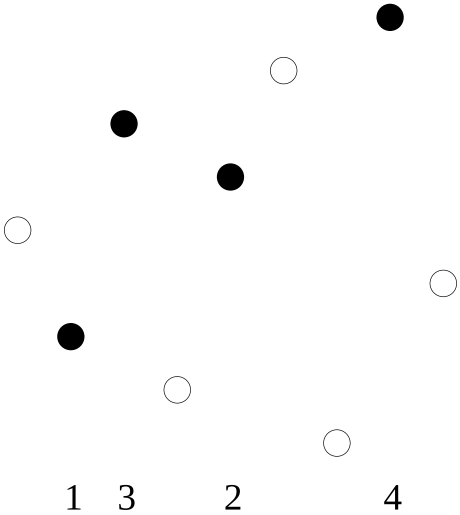

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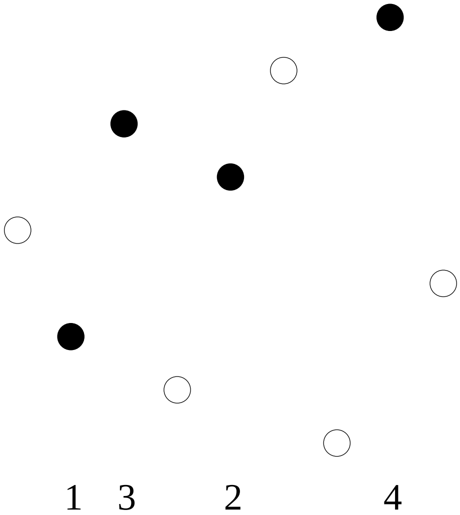

$$1324 \preceq 537268194$$

Definition

If we construe permutations as sequences, then the *involvement order* on permutations is defined by:

$$\alpha \preceq \beta \quad \text{iff} \quad \left\{ \begin{array}{l} \beta \text{ contains a subsequence} \\ \text{whose terms are in the same} \\ \text{relative order as those of } \alpha \end{array} \right\}$$

A *permutation class*, $\mathcal{C}$ is a down-closed set for $\preceq$. Its *basis* X consists of the $\preceq$-minimal permutations *not in* $\mathcal{C}$, and then:

$$\mathcal{C} = \mathrm{Av}(X) \stackrel{\text{def}}{=} \{\beta : \forall \alpha \in X \; \alpha \not\preceq \beta\}.$$

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- The idea is, where possible, to look for general results rather than specific *ad hoc* examples.
- But, we are not unhappy with solving specific examples if they are interesting!

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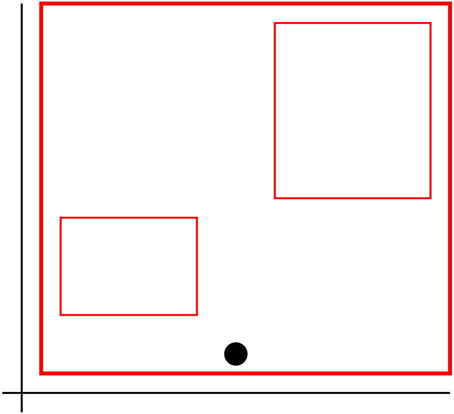
- Time for some examples!

Av(312)



$\mathrm{Av}(312) = \{\alpha\, 1\, \beta \;:\; \alpha < \beta$ both avoiding 312$\}$

But also $\mathrm{Av}(312)$ is . . .

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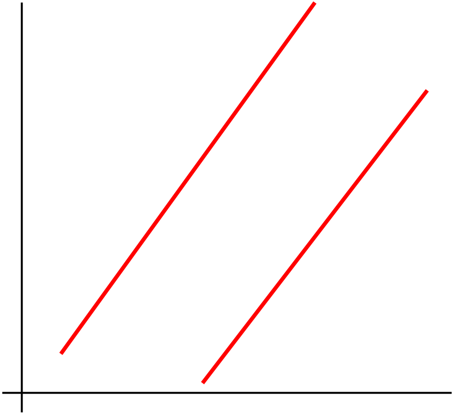
- Enumerated by the Catalan numbers.

Av(321)



$\mathrm{Av}(321) = \{\pi \ : \ \pi$ a union of two monotone sequences$\}$

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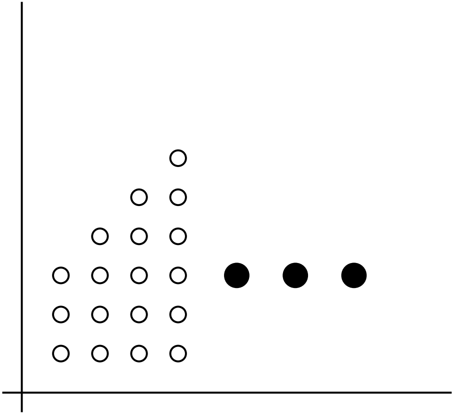
Av(4123, 4132, 4213, 4231, 4312, 4321)



$\pi \in \mathrm{Av}(4\bullet\bullet\bullet)$ if each symbol of π is among the three smallest of its suffix.

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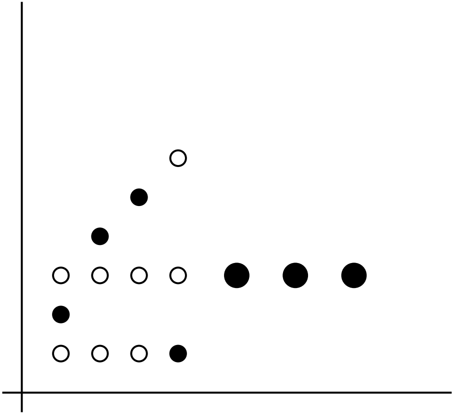

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But also $\mathrm{Av}(4 \bullet \bullet\bullet)$ is ...

- The set of permutations that can be produced from the input $123\cdots n$ by a buffer capable of holding no more than three items at a time.
- Enumerated by $3^{n-2} \times 2 \times 1$.
- Easily encoded over a three symbol alphabet.

$\mathcal{C}$ is Finite

If, and only if, for some n and k:

$$\begin{aligned} 123\cdots n &\notin \mathcal{C} \\ k(k-1)\cdots 321 &\notin \mathcal{C}. \end{aligned}$$

(or, more briefly, for some n, neither $123\cdots n$ nor $n\cdots 321$ is in $\mathcal{C}$.)

(Erdős-Szekeres).

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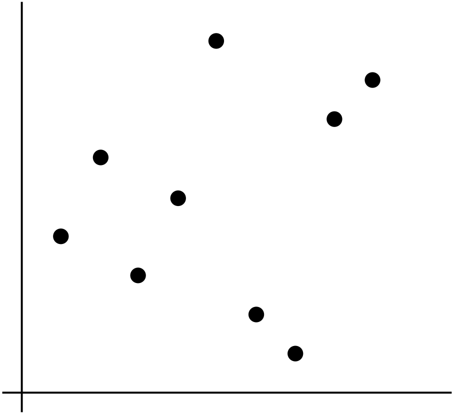
- Conditions on the basis to give structure.

Block Decomposition



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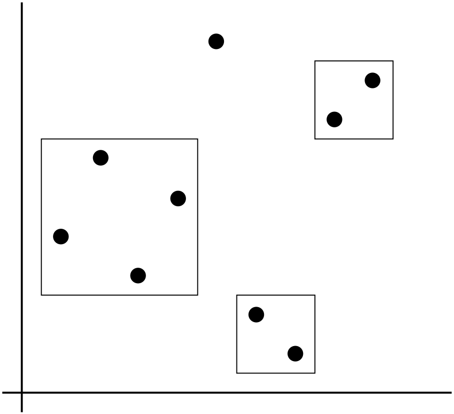

$$463592178 = 2413[2413, 1, 21, 12]$$

Simple Permutations

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- The first few: $12, 21, 2413, 3142, 24153, \ldots$
- Every permutation is the *inflation* of a unique simple permutation, called its *skeleton*. This is called its *block decomposition*. The blocks are also uniquely determined if the skeleton is not 12 or 21. In that case we can enforce uniqueness by requiring that the first block not be so decomposable (so $21354 = 12[21, 132]$.)

Random Question

The total number of simple permutations of length n is asymptotically $n!/e^2$, i.e. a positive proportion of all permutations are simple. But, it appears that in any infinite class $\mathcal{C}$, the simple elements of $\mathcal{C}$ have density 0.

Is that true? If so, why?

Wreath Closed Classes

- A class, $\mathcal{C}$, is *wreath closed* if, whenever σ (of length k) and $\pi_1, \pi_2, \ldots, \pi_k$ (of arbitrary lengths) are in $\mathcal{C}$, then so is $\sigma[\pi_1, \pi_2, \ldots, \pi_k]$.

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- Equivalently, every basis element of $\mathcal{C}$ is simple.
- Equivalently, $\mathcal{C}$ is the closure of a downward closed set of simple permutations under inflation.

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If $\mathcal{C}$ is a class with finitely many simple permutations then:

- $\mathcal{C}$ has a finite basis,
- $\mathcal{C}$ has an algebraic generating function
- and much much more (Brignall, Huczynska, Vatter)

Furthermore

There is an effective procedure, given a finite basis X to determine whether or not $\mathrm{Av}(X)$ contains only finitely many simple permutations (Brignall, Ruškuc, Vatter).

This is based on the existence of certain unavoidable structures in large simple permutations, not unlike the Erdős-Szekeres characterization of finite classes.

Are classes with finitely many simples then “finis”?

Separable Permutations

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- $\mathcal{S} = \mathrm{Av}(2413, 3142)$ is the wreath closure of 12 and 21. It is called the class of *separable* permutations, and is enumerated by the large Schroeder numbers.
- The degree over $\mathbb{Q}(t)$ of the generating function of any subclass of $\mathcal{S}$ is a power of 2.
- If $\pi_1 = 132$, and

$$\pi_{n+1} = \begin{cases} 12[1, \pi_n] & n \text{ even,} \\ 21[1, \pi_n] & n \text{ odd.} \end{cases}$$

(so: 132, 4132, 15243, 615243, $\dots$) then the degree of the generating function of $\mathrm{Av}(2413, 3142, \pi_n)$ over $\mathbb{Q}(t)$ is *precisely* 2^n.

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- What can be said about the set of growth rates of separable classes?

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- Alternatively, we can use “histoires de Laguerre” (X. Viennot), restricted in various ways.
- Other examples include the W-classes (where the number of monotone runs is bounded.)

A Metatheorem

Suppose that an encoding of a class $\mathcal{C}$ over a finite alphabet Σ is such that the relation:

$$\sigma \preceq \pi$$

(for $\sigma, \pi \in \mathcal{C}$) is accepted by a finite state transducer.

Then a subclass of $\mathcal{C}$ is a regular set in Σ^* if and only if its basis (relative to $\mathcal{C}$) is regular.

In particular, . . .

Rank Bounded Classes

- The required transducer commits in advance to which of the k smallest remaining symbols must be deleted.
- Think of its states as encoded by bit strings $b_1 b_2 \ldots b_k$ with $b_j = 1$ meaning "I promise to delete the j-th smallest remaining symbol."
- To process an input symbol, check first if it is to be deleted. If so, output nothing; if not, output its value minus the number of smaller items to be deleted. Then, in either case, eliminate its bit from the string, and add a new final bit of your choice.
- Do some minor tinkering to handle end cases.

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- But, every finitely based subclass of $\mathcal{B}_3$ has a rational generating function.
- (A new question?) Is there a subclass of $\mathcal{B}_3$ with an algebraic, but not rational, generating function?

Other sorts of structure

- A class, $\mathcal{C}$, is *atomic* (a.k.a. “has the joint embedding property”) if, for all $\alpha, \beta \in \mathcal{C}$, there exists $\pi \in \mathcal{C}$ with $\alpha, \beta \preceq \pi$.

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- This is the model theoretic approach and is very much in its infancy.

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- Every proper subclass of the first has a rational generating function.
- Not much (really, next to nothing) is known about the behaviour of even finitely based subclasses of the second.

The Frontier

Is wide open:

- What about $\mathrm{Av}(4231)$?
- What about "simple" machines (two stacks in series, for example.)
- What about detailed understanding of $\mathcal{S}$? (prediction of degree of algebraicity without computation; characterization of growth rates, $\ldots$)
- How well can we "approximate" arbitrary classes with ones having structure?