Halfway up the Stairs

Michael Albert
malbert@cs.otago.ac.nz

Permutation Patterns, 2008

UNIVERSITY of OTAGO
SAPERE AUDE
Te Whare Wānanga o Otāgo
NEW ZEALAND

Credit Department

- Most of this talk reports on joint work with Mike Atkinson, Robert Brignall, Nik Ruškuc, Rebecca Smith and Julian West.
- The postscript reports on joint work with Vince Vatter.

Context

All the usual stuff:

- A *permutation class*, $\mathcal{C}$ is a set of permutations closed downwards under involvement.
- The *growth rate* of $\mathcal{C}$ is:

$$\limsup_{n \to \infty} |\mathcal{C} \cap \mathcal{S}_n|^{1/n}.$$

- For permutations α and β, their sum $\alpha \oplus \beta$ has pattern α, below and followed by pattern β.

An Intriguing Observation

Let

$$\delta_t = t\,(t-1)\,(t-2)\,\cdots\,3\,2\,1$$

Suppose that π avoids δ_{k+1}, involves $\alpha \oplus 1 \oplus \beta$, but avoids $\alpha \oplus 1 \oplus 1 \oplus \beta$.

Then, there can be at most k elements in π that play the rôle of 1 in an embedding of $\alpha \oplus 1 \oplus \beta$.

An Intriguing Observation

Let

$$\delta_t = t\,(t-1)\,(t-2)\,\cdots\,3\,2\,1$$

Suppose that π avoids δ_{k+1}, involves $\alpha \oplus 1 \oplus \beta$, but avoids $\alpha \oplus 1 \oplus 1 \oplus \beta$.

Then, there can be at most k elements in π that play the rôle of 1 in an embedding of $\alpha \oplus 1 \oplus \beta$.

Because, the second condition forces such elements to form a descending chain.

An Intriguing Observation

Let

$$\delta_t = t\,(t-1)\,(t-2)\,\cdots\,3\,2\,1$$

Suppose that π avoids δ_{k+1}, involves $\alpha \oplus 1 \oplus \beta$, but avoids $\alpha \oplus 1 \oplus 1 \oplus \beta$.

Then, there can be at most k elements in π that play the rôle of 1 in an embedding of $\alpha \oplus 1 \oplus \beta$.

Because, the second condition forces such elements to form a descending chain.

Therefore, the growth rates of the classes Av($\delta_{k+1}, \alpha \oplus 1 \oplus \beta$) and Av($\delta_{k+1}, \alpha \oplus 1 \oplus 1 \oplus \beta$) are the same.

So obviously …

Is it true that the growth rates of Av($\delta_{k+1}, \alpha \oplus 1 \oplus \beta$) and Av($\delta_{k+1}, \alpha \oplus \beta$) are the same?

So obviously . . .

Is it true that the growth rates of $\text{Av}(\delta_{k+1}, \alpha \oplus 1 \oplus \beta)$ and $\text{Av}(\delta_{k+1}, \alpha \oplus \beta)$ are the same?

I don't know.

So obviously . . .

Is it true that the growth rates of Av$(\delta_{k+1}, \alpha \oplus 1 \oplus \beta)$ and Av$(\delta_{k+1}, \alpha \oplus \beta)$ are the same?

I don't know.

But a generalization of this *is* true, for $k = 2$.

Rank and Rigidity

- The *rank* of x in a permutation π is the largest t such that x is the maximum of some δ_t pattern.

Rank and Rigidity

- The *rank* of x in a permutation π is the largest t such that x is the maximum of some δ_t pattern.
- A permutation, π, is *k-rigid* if it avoids δ_{k+1}, and every $x \in \pi$ belongs to some δ_k.

Obvious Observations

- If π avoids δ_{k+1} and x occurs in some δ_k, then the position of x in every δ_k that it occurs in is the same.

Obvious Observations

- If π avoids δ_{k+1} and x occurs in some δ_k, then the position of x in every δ_k that it occurs in is the same.
- If α is k-rigid and π avoids δ_{k+1}, then any embedding of α in π must preserve rank.

Obvious Observations

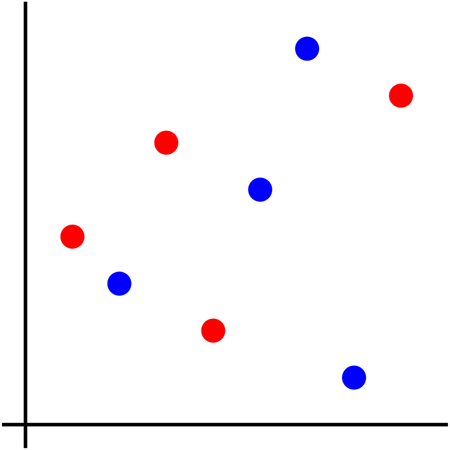
- If π avoids δ_{k+1} and x occurs in some δ_k, then the position of x in every δ_k that it occurs in is the same.
- If α is k-rigid and π avoids δ_{k+1}, then any embedding of α in π must preserve rank.
- In particular, if $p, q \in \pi$ are both the images of $a \in \alpha$ (under two different embeddings), then the pattern of $\{p, q\}$ is 1 or 12.

A Lattice of Embeddings

Theorem

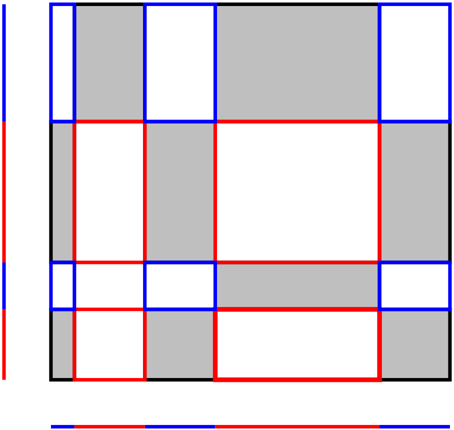
Let α be a k-rigid permutation and π avoid δ_{k+1}. The embeddings of α in π form a distributive lattice under pointwise minimum and maximum.

Merge



43625817 is a merge of 2314 and 2341.

Bounded Merge



In a *bounded merge* the number of red (or blue) intervals (by both position and value) is bounded in advance.

Growth Rate of Merged Classes

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of growth rates a and b respectively.

- The growth rate of $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is at most

$$a + b + 2\sqrt{ab}$$

  (equality holds if either growth rate is a limit.)
- For any bound B, the growth rate of $\mathcal{M}_B(\mathcal{A}, \mathcal{B})$ is

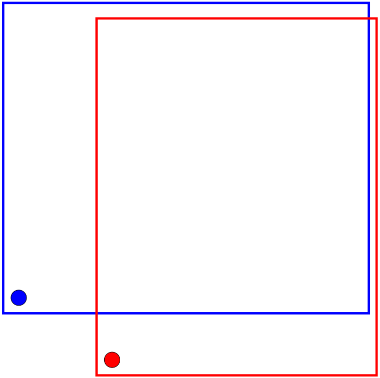
$$\max(a, b).$$

The Grand Strategy

To show that the growth rate of Av$(321, \beta)$ and Av$(321, 1 \oplus \beta)$ are the same, show that any

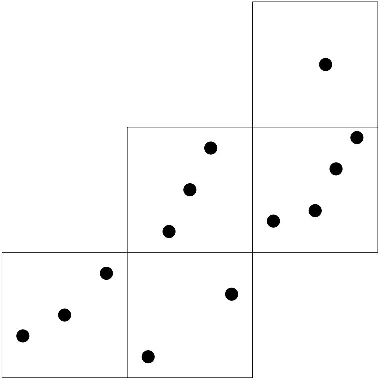
$$\pi \in \text{Av}(321, 1 \oplus \beta)$$

must be the bounded merge of two permutations λ and ρ, each beginning with their minimum element.



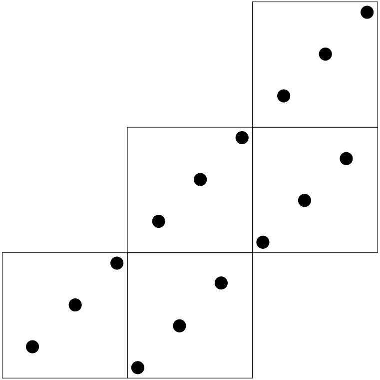
Staircases

Any $\pi \in \mathrm{Av}(321)$ can be decomposed as a *staircase*.



Generic Staircases

In a *generic staircase*, the steps interlock in the obvious way (below, 5 steps of size 3, so a $(5, 3)$-generic staircase.)



Two Important Observations

- Every β in Av(321) embeds in a (k, s)-generic staircase for some k and s.
- For every (k, s) there is a B such that any $\pi \in$ Av(321) either contains a (k, s)-generic staircase, or is the B-bounded merge of two permutations each beginning with its minimum.

So What?

That completes the grand plan for the case $\text{Av}(321, \beta)$ versus $\text{Av}(321, 1 \oplus \beta)$.

So What?

That completes the grand plan for the case Av$(321, \beta)$ versus Av$(321, 1 \oplus \beta)$.

- Consider the latter (and larger) class. Take a permutation π in it.

So What?

That completes the grand plan for the case Av$(321, \beta)$ versus Av$(321, 1 \oplus \beta)$.

- Consider the latter (and larger) class. Take a permutation π in it.
- Choose (k, s) such that $1 \oplus \beta$ is involved in the (k, s)-generic staircase.

So What?

That completes the grand plan for the case $\mathrm{Av}(321, \beta)$ versus $\mathrm{Av}(321, 1 \oplus \beta)$.

- Consider the latter (and larger) class. Take a permutation π in it.
- Choose (k, s) such that $1 \oplus \beta$ is involved in the (k, s)-generic staircase.
- Since π cannot involve this staircase, it is a bounded merge of two permutations each beginning with their minimum.

So What?

That completes the grand plan for the case $\text{Av}(321, \beta)$ versus $\text{Av}(321, 1 \oplus \beta)$.

- Consider the latter (and larger) class. Take a permutation π in it.
- Choose (k, s) such that $1 \oplus \beta$ is involved in the (k, s)-generic staircase.
- Since π cannot involve this staircase, it is a bounded merge of two permutations each beginning with their minimum.
- But, then the rest of these permutations avoid β, i.e.

$$\text{Av}(321, 1 \oplus \beta) \subseteq \mathcal{M}_B(1 \oplus \text{Av}(321, \beta), 1 \oplus \text{Av}(321, \beta))$$

and we're done.

Reductions

In general a permutation in Av(321) can be written in the form:

$$1^{m_0} \oplus \alpha_1 \oplus 1^{m_1} \oplus \alpha_2 \oplus \cdots \oplus \alpha_t \oplus 1^{m_t}$$

where the α_i are rigid. Define its *reduced form* to be:

$$\alpha_1 \oplus \alpha_2 \oplus \alpha_t$$

(which is also the maximum rigid permutation that it contains).

The Full Theorem

Theorem

Let X *be any subset of* $\mathrm{Av}(321)$*, not containing an increasing permutation. Let* X' *be the set of reduced forms of all the elements of* X*. Then, the growth rates of* $\mathrm{Av}(321, X)$ *and* $\mathrm{Av}(321, X')$ *are the same.*

The Full Theorem

Theorem

Let X be any subset of Av(321)*, not containing an increasing permutation. Let X' be the set of reduced forms of all the elements of X. Then, the growth rates of* Av$(321, X)$ *and* Av$(321, X')$ *are the same.*

The required extensions to the proof:

- To eliminate a 1 from Av$(321, \alpha \oplus 1 \oplus \beta)$ when α is rigid.
- Start with a leftmost/bottommost embedding of α.
- Show that the bounded merge in the $1 \oplus \beta$ avoiding part above and to the right of it can be glued on to the remainder of the permutation, representing it as a bounded merge of two permutations each of which, after the deletion of a single point, avoids $\alpha \oplus \beta$.
- Induction.

Postscript

The previous discussion brings to attention the class of those permutations avoiding 321 that can be decomposed into a staircase of at most k steps (fixing k).

Postscript

The previous discussion brings to attention the class of those permutations avoiding 321 that can be decomposed into a staircase of at most k steps (fixing k).

How many of them are there?

Postscript

The previous discussion brings to attention the class of those permutations avoiding 321 that can be decomposed into a staircase of at most k steps (fixing k).

How many of them are there?

I don't know.

Postscript

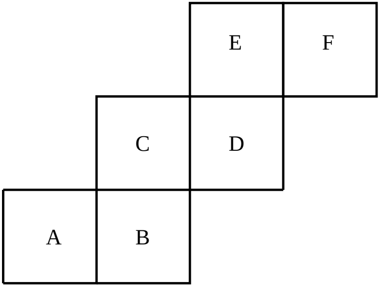
The previous discussion brings to attention the class of those permutations avoiding 321 that can be decomposed into a staircase of at most k steps (fixing k).

How many of them are there?

I don't know.

But, I can tell you about the growth rate of this class.

Enumerative Observations



In this picture of a staircase, the number of permutations where the boxes have the indicated sizes is (almost exactly)

$$\binom{A+B}{A}\binom{B+C}{B}\binom{C+D}{C}\binom{D+E}{D}\binom{E+F}{E}$$

How Big is a Staircase?

After visits from Mr Stirling and Comte Lagrange, and the assistance of *Maple*, together with a certain amount of more or less clever rearrangement, the optimization problem arising from the observations above can be solved.

Theorem

The growth rate of a monotone staircase grid class with k *cells is* $1 + t$ *where* t *is the largest positive solution of*

$$0 = t - \cfrac{1}{t - 1 - \cfrac{1}{t - 1 - \cfrac{1}{\cdots - \cfrac{1}{t - 1}}}}$$

if k *is even, where* $t - 1$ *occurs* $k/2$ *times.*

Loose Ends

There are lots!

Loose Ends

There are lots!

- Does the same result hold within Av(δ_{k+1}) for $k > 2$?

Loose Ends

There are lots!

- Does the same result hold within Av(δ_{k+1}) for $k > 2$?
- What can be said about the lattices of embeddings of a k-rigid permutation into permutations avoiding δ_{k+1}?

Loose Ends

There are lots!

- Does the same result hold within Av(δ_{k+1}) for $k > 2$?
- What can be said about the lattices of embeddings of a k-rigid permutation into permutations avoiding δ_{k+1}?
- Can we count k-rigid permutations?

Loose Ends

There are lots!

- Does the same result hold within Av(δ_{k+1}) for $k > 2$?
- What can be said about the lattices of embeddings of a k-rigid permutation into permutations avoiding δ_{k+1}?
- Can we count k-rigid permutations?
- Can we count (exactly) permutations in the staircase classes?

Loose Ends

There are lots!

- Does the same result hold within Av(δ_{k+1}) for $k > 2$?
- What can be said about the lattices of embeddings of a k-rigid permutation into permutations avoiding δ_{k+1}?
- Can we count k-rigid permutations?
- Can we count (exactly) permutations in the staircase classes?
- ...