Bounds for the Growth Rate of Meander Numbers

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FPSAC, Vancouver, 2004

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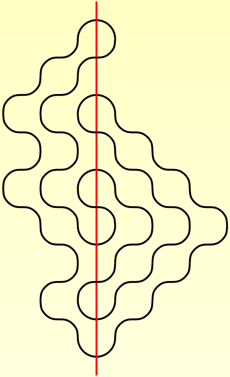
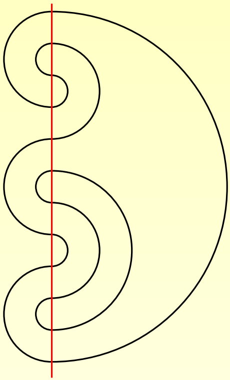


Topological Meanders

- Informally a meander is (the equivalence class of) an image of S^1 in $\mathbb{R}^2$ which crosses the vertical axis at least once, and is never tangent to that axis.
- Two such images are equivalent if they are homotopic with respect to a homotopy $h : I \times \mathbb{R}^2 \to \mathbb{R}^2$ which fixes the y-axis (as a set).

Combinatorial Meanders

- The arches on either side of the y-axis provide a natural identification between meanders and pairs of bracket sequences of equal length.
- Not all such pairs represent meanders.



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- Exact answers known for small(ish) *n* based on a number of algorithmic approaches.
- Conjectured asymptotic forms based on correspondences with certain field theories.
- We seek upper and lower bounds on the exponential part of the asymptotic form.

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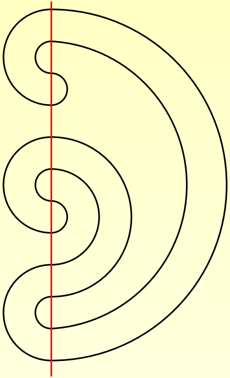
- We show that:

$$11.380 \leq M \leq 12.901.$$

- Previous best bounds approximately $10 \leq M \leq 13$ due to R. Stanley (lower bound, 1995) and J. Reeds and L. Shepp (1999).

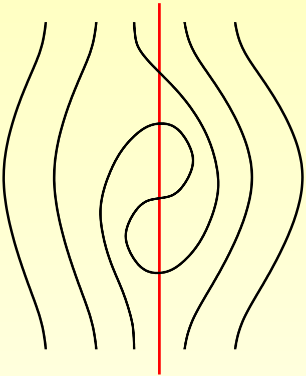
Encoding

- Think of the evolution of a meander as we progress upwards on the vertical axis.
- Each crossing represents an *event* and the meander is characterized by this sequence of events.
- There are four event types U, R, L and D.
- In the example to the right, the meander can be represented by the sequence $UURRULDDULDD$.



Environment

- It is sometimes helpful to imagine an *environment* of segments on either side of the vertical axis.
- Some factors such as *URDL* have no effect on the environment (see diagram).
- These can be freely substituted into meander words *in almost all contexts*, yielding new meander words.



Shifts and Jumps

- A *shift* is a word whose effect on the environment is the same as R^k or L^k for some k. Its *displacement* is k or $-k$ respectively.
- A *jump* is a shift, none of whose prefixes are shifts.
- A *primitive jump* is a jump all of whose shift factors are of the form R^* or L^*.
- The sequence *UURURDDD* is a jump (of displacement 2). It is not primitive due to the factor *URD*. The corresponding primitive form is *UURRDD*.

Lower Bounds

- From a balanced finite set of primitive jumps we can build a language of shifts by recursive subsitution.
- The words of length $\approx 2n$ in this set will very rarely be extremely unbalanced in terms of R's and L's. So almost all form legitimate continuations in a meander environment formed by a prefix such as:

$$U^{2n^{3/4}} R^{n^{3/4}}.$$

- Thus the rate of growth for such shift languages is dominated by the rate of growth for the meander language, thus providing us with a lower bound computation.

Fundamental Observation

- Certain words are forbidden as factors within a long meander word.
- These words are those which demonstrably create a submeander in any context in which they occur. The most obvious one is UD, another is $URULLD$.
- The full meander language is a sublanguage of the language avoiding these forbidden factors.

Application of the Cluster Method

- The *cluster method* (Goulden and Jackson) gives an equation for the generating function of the language of words over an alphabet, avoiding a finite set of forbidden factors.
- Generalized by Zeilberger to handle certain infinite families.
- The underlying equation applies to an arbitrary collection of forbidden factors, and can be used in our setting.

Iterative Schemes

- Both the upper bound and lower bound computations provide equations for the radius of convergence of the associated generating function that cannot be solved exactly.
- Since the relationships satisfied by the generating functions are essentially fixed point equations, we can approximate them iteratively, specializing to a fixed value of the formal variable at the outset.

Lower bounds

- For lower bounds we can use convergence or divergence as a criterion for determining an approximation to the radius of convergence, and hence the associated growth rate.
- Using all primitive jumps of length at most 24 in the equations gives $11.380 \leq M$.

Upper bounds

- For upper bounds, we consider the positivity or negativity of the denominator in an expression for the generating function.
- Using all submeander words of length at most 16 as the forbidden factors gives $M \leq 12.901$.

Conclusions

- Further improvements in the bounds could be obtained by using larger sets of primitive jumps or submeander factors.
- Simplistic approximations suggest that these would yield, $11.6 \leq M \leq 12.8$ however, there is some evidence that the lower bound improvements may be larger than this.