Pattern avoidance sets and infinite permutations

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Outline of talk

Review of terminology of closed sets

Examples of their origin

Atomic sets and a new view of closed sets

Main theorem and its proof

Pattern avoidance sets

- Sets $\mathcal{X}$ of permutations that avoid a fixed set of permutation patterns

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- To compute the number of permutations of each length we have to work out *structural properties* of $\mathcal{X}$

Where closed sets come from

- An explicit set of permutations to avoid
- Permutations generated by stacks and other data structures
- Token-passing networks [Atkinson et al., 1997]
- *Ad hoc* combinatorial constructions
- Subpermutations of some infinite bijection

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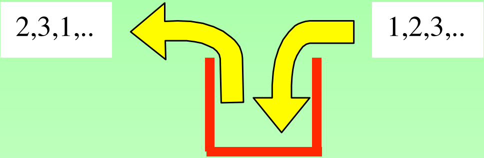

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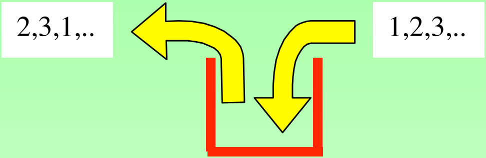

Characterized by avoiding 312

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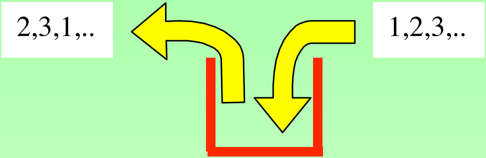

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$\frac{\binom{2n}{n}}{n+1}$ permutations of length n

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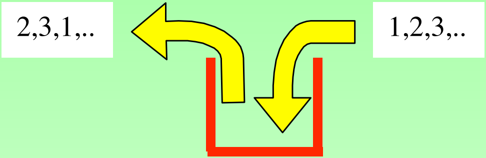

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For a large number of variations see Miklos Bóna's survey [Bóna, 2003].

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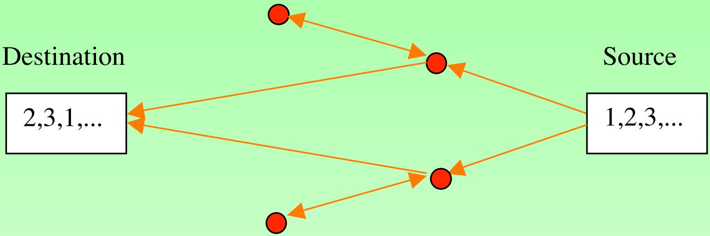

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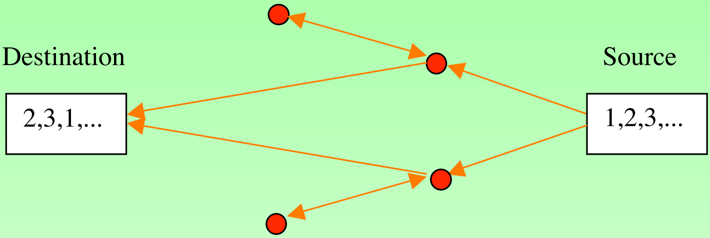

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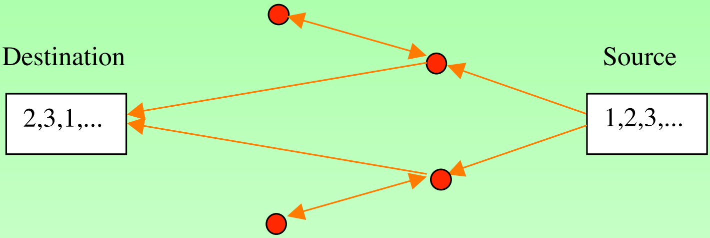
Token passing networks



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This one cannot be defined by a finite number of restrictions

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- Many other examples

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Infinite bijections

Define subsets of $\Re$

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\begin{aligned}
A &= \{1 - 1/2^i, 2 - 1/2^i \mid i = 1, 2, \ldots\} \\
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and let $\pi : A \longrightarrow B$ be defined by

$$\pi(x) = \begin{cases} 2i - 1 & \text{if } x = 1 - 1/2^i \\ 2i & \text{if } x = 2 - 1/2^i \end{cases}$$

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The set $\mathrm{Sub}(\pi)$ of (finite) subpermutations of π is a closed set (in this case defined by the restrictions $321, 2143, 3142$)

A structure theory?

Can we devise a structure theory that says sensible things about closed sets but does not depend on how they are presented?

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- If we understand atomic sets all we need do is take unions

The structure of atomic sets

Theorem

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$\Longleftarrow$: Suppose $\mathcal{X} = \mathrm{Sub}(\pi)$ but $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$. Choose $\eta \in \mathcal{Y} \setminus \mathcal{Z}$ and $\zeta \in \mathcal{Z} \setminus \mathcal{Y}$. Then η, ζ are represented in π as subsequences. Their union represents a permutation $\theta \in \mathcal{X}$ containing both η and ζ. So $\theta \in \mathcal{Y} \Longrightarrow \zeta \in \mathcal{Y}$ etc.

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- Therefore consider the simplest case where the order type of A and B is that of $\mathbb{N}$.
- A closed set is *natural* if it has the form $\mathrm{Sub}(\pi)$ where $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection

A supply of natural sets

- Many closed sets have the *sum-complete* property

$$\sigma, \tau \in \mathcal{S} \Longrightarrow \sigma \oplus \tau \in \mathcal{S}$$

Picture

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Picture

- If $\mathcal{S}$ is closed and sum-complete and γ is any (finite) permutation then $\mathrm{Sub}(\gamma) \oplus \mathcal{S}$ is natural
  - List the permutations of $\mathcal{S}$ as $\sigma_1, \sigma_2, \ldots$ and put

$$\pi = \gamma \oplus \sigma_1 \oplus \sigma_2 \cdots$$

Main result

Theorem

If $\mathcal{X}$ is natural and finitely based then either

- $\mathcal{X} = \mathrm{Sub}(\gamma) \oplus \mathcal{S}$ *where γ is finite and $\mathcal{S}$ is sum-complete and finitely based, or*
- π *is periodic from some point on; i.e. there exist integers N and $p > 0$ such that, for all $n > N$, $\pi(n+p) = \pi(n) + p$. In this case π is determined by $\mathcal{X}$ uniquely.*

Periodic example

Finite basis is necessary

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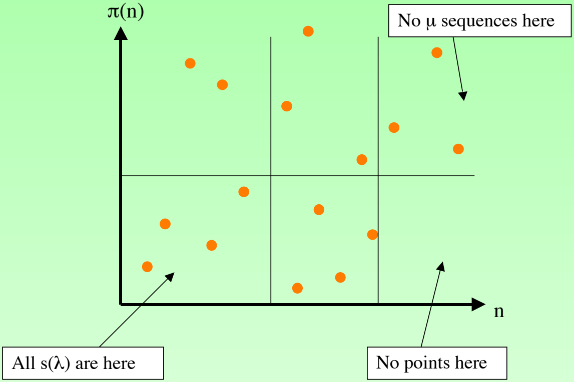
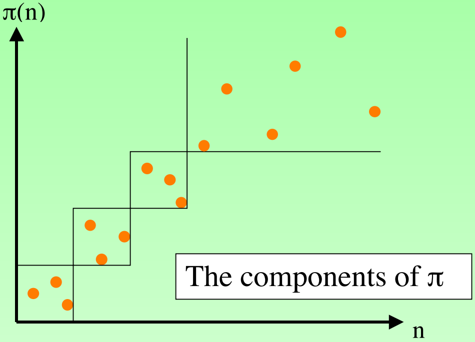
- From some point on in π we are beyond all the $s(\lambda)$s (finite basis!)
- From some further point on we only have values larger than those occurring in the $s(\lambda)$s (no limit points!)
- From this point in π no subsequences isomorphic to a μ sequence occur

The graph of π



The division into cases



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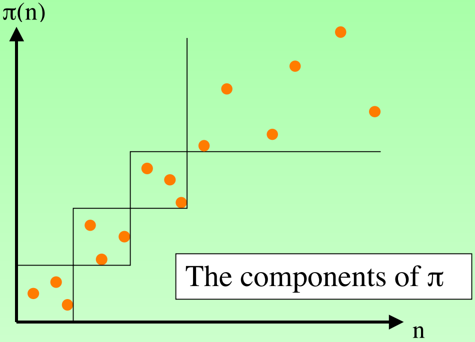

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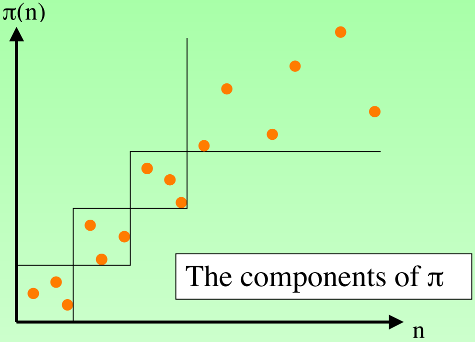

- All components from some point on are free of μ-sequences; this gives $\mathrm{Sub}(\gamma) \oplus \mathcal{S}$
- Finitely many components and the final component contains a μ-sequence; this gives π periodic

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- Only exceptions are when π is periodic
- What can we say about $\mathrm{Sub}(\pi)$ when π is periodic?

Properties of $\mathrm{Sub}(\pi)$ for periodic π

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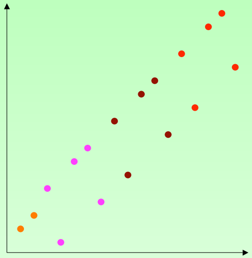
- Basis of $\mathrm{Sub}(\pi)$ can be computed even if it is not finite
- $\mathrm{Sub}(\pi)$ can be enumerated algorithmically
- Generating function of $\mathrm{Sub}(\pi)$ is rational

Not every periodic $\mathrm{Sub}(\pi)$ is finitely based

- If

$$\pi = 2\ 3\ \{5\ 1\ 7\ 8\ 4\}\ \{10\ 6\ 12\ 13\ 9\}\ \{15\ 11\ 17\ 18\ 14\}\ \dots$$

then $\mathrm{Sub}(\pi)$ is not finitely based.



Questions?

References

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Theoretical Computer Science 306 (2003), 85–100.

M. D. Atkinson, D. H. Tulley, M. J. Livesey
Permutations generated by token passing in graphs
Theoretical Computer Science 178 (1997), 103–118.

M. Bóna
A Survey of Stack-Sorting Disciplines
Electronic J. Combin. 9(2) (2003), Paper A1.

The finite basis assumption is necessary

- Let

$$\pi = 3\ 2\ 5\ 1\ [7, 8]\ 4\ [10, 12]\ 6\ [14, 17]\ 9\ [19, 23]\ 13\ [25, 30]\ 18 \ldots$$

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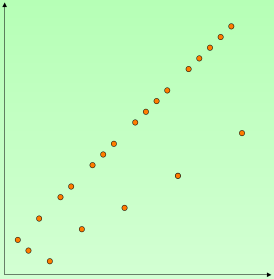

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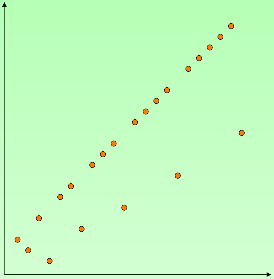
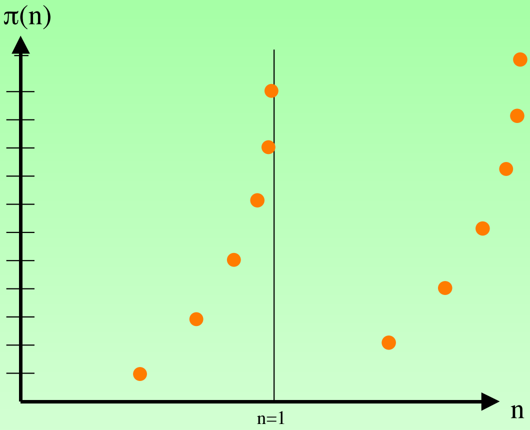
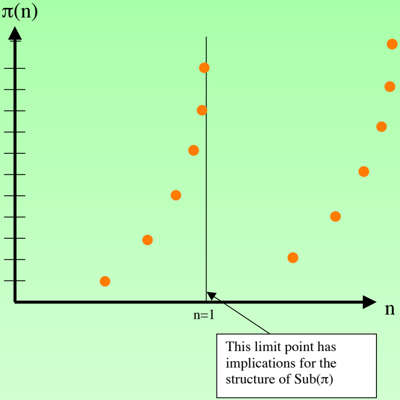


- π is not periodic and $\mathrm{Sub}(\pi)$ is not of the form "$\mathrm{Sub}(\gamma) \oplus \mathcal{S}$ where $\mathcal{S}$ is sum-complete".

Back to theorem



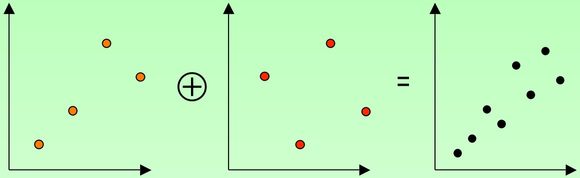
◂ Back


π(n)
n
n=1
This limit point has implications for the structure of Sub(π)

Back

The $\oplus$ operation

$$1243 \oplus 3142 = 12437586$$



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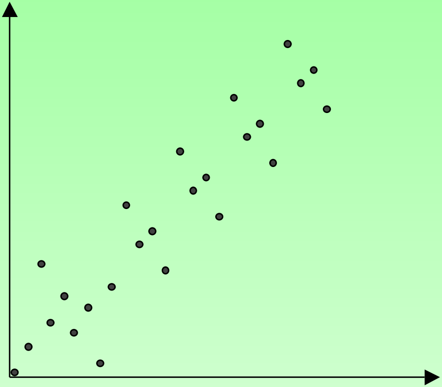

Periodic π

◂ Back to theorem