Sorting classes, the weak and strong orders

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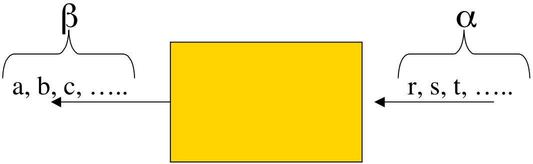
Outline of talk

Permuting machines



- The output β is a (non-deterministic) rearrangement of the input α
- The names of the input items are immaterial; use names $1, 2, \ldots$
- If some input items are omitted the machine can rearrange the remaining ones as they were arranged in the original

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- Example: 312 $\mathcal{I}$ 7531462
- A *permutation class* is a set of permutations closed downwards in the $\mathcal{I}$ order
- The set of sortable inputs of a permuting machine is always a permutation class (it is $\mathrm{av}(\pi_1, \pi_2 \ldots)$)

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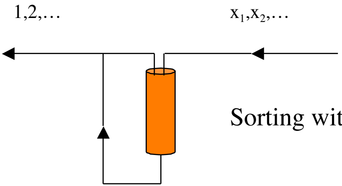
Example 1 – from Knuth [1]



Sorting with a tube

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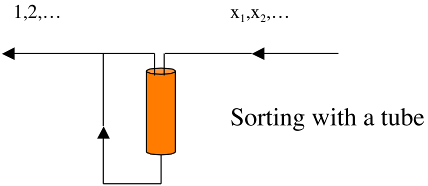

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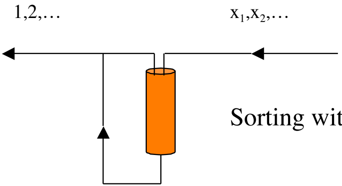

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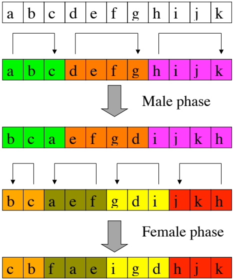
- If there are s_n sortable permutations of length n then

$$\sum_{n=0}^{\infty} s_n x^n = \frac{1}{2}(3 - x - \sqrt{1 - 6x + x^2})$$

Example 2: the male-female sorting machine

Operates in two phases: a *male* phase then a *female* phase.



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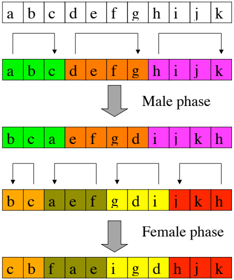

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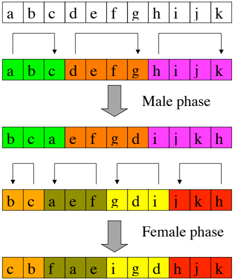

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If there are t_n sortable permutations of length n then

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- Enter the weak and strong (Bruhat) orders.

The weak and strong orders

- The *weak order* $\mathcal{W}$ is the transitive closure of the relations

$$\lambda ab\mu \ \mathcal{W} \ \lambda ba\mu \text{ where } b > a$$

Example: 41523 $\mathcal{W}$ 45123 $\mathcal{W}$ 45132 $\mathcal{W}$ 45312

- The *strong order* $\mathcal{S}$ is the transitive closure of the relations

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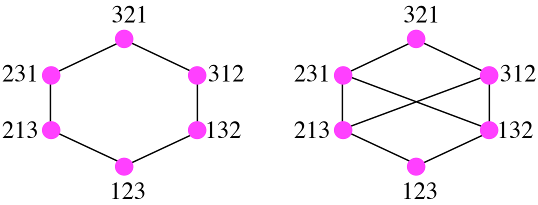
The weak and strong orders on S_3



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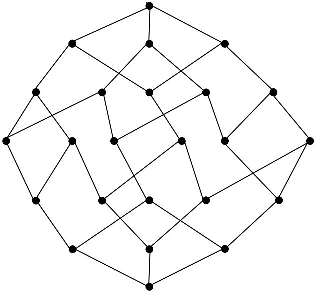

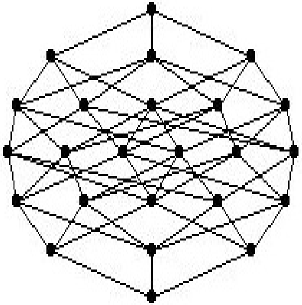

Weak and Strong Sorting Machines

- Weak sorting machine: a permuting machine that, if it can sort α, can also sort any β with $\beta \; \mathcal{W} \; \alpha$.
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- The set of sortable permutations for a sorting machine is a *sorting class*

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- Example: The permutations sortable by the male-female sorting machine

Extending the pattern containment order

Weak and strong sorting classes are down-ideals in the partial orders $\{\mathcal{I} \cup \mathcal{W}\}^*$ and $\{\mathcal{I} \cup \mathcal{S}\}^*$ (respectively). What do these orders look like?

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1. $\{\mathcal{I} \cup \mathcal{W}\}^* = \mathcal{I}\mathcal{W} = \mathcal{W}\mathcal{I}$, *and*
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We can study weak and strong sorting classes by their forbidden patterns in these orders, imitating ordinary pattern class studies.

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- What is the strong sorting class defined by avoiding (in the $\{\mathcal{I} \cup \mathcal{S}\}^*$ sense) the permutation 3421? It is $\mathrm{av}(3421, 4321, 34512, 43512, 35412, 53412)$.

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- What is the strong sorting class defined by avoiding (in the $\{\mathcal{I} \cup \mathcal{S}\}^*$ sense) the permutation 3421? It is $\mathrm{av}(3421, 4321, 34512, 43512, 35412, 53412)$.
- Do weak and strong sorting classes have special structural properties that help us (e.g.) to solve their enumeration problems? YES for strong sorting classes, not so clear for weak sorting classes.

Weak sorting classes

Most results stem from $\{\mathcal{I} \cup \mathcal{W}\}^* = \mathcal{I}\mathcal{W} = \mathcal{W}\mathcal{I}$.
For example:

Lemma

$\mathrm{av}(\pi_1, \pi_2, \ldots, \pi_k)$ *is a weak sorting class if and only if every permutation above any* π_i *(in the weak order) contains one of the* π_j *as a pattern.*

E.g. $\mathrm{av}(321, 3124)$ is not a weak sorting class because $3124\ \mathcal{W}\ 3142$ but 3142 contains neither 321 or 3124 as a pattern.

Strong sorting classes

- The theory of strong sorting classes is quite different because $\mathcal{IS} \neq \mathcal{SI}$.
- Example: 321 $\mathcal{I}$ 3214 $\mathcal{S}$ 3412 but no δ with 321 $\mathcal{S}$ δ $\mathcal{I}$ 3412.
- Despite this the structure of strong sorting classes is much more constrained than the structure of weak sorting classes.

The classes $\mathcal{B}(r, s)$ – see also Mansour and Vainshtein [2]

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$\mathcal{B}(r, s)$ is the class of all permutations which do not have a subsequence of $r + s$ elements the first r all larger than the last s. This is a strong sorting class.

7 4 **12** 8 5 **9** 2 11 **6** 10 **1** **3**

Not in B(3,3)

The role of $\mathcal{B}(r, s)$

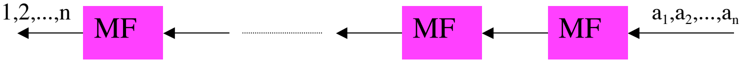
Theorem

If $\mathcal{X}$ is a strong sorting class not containing all permutations then $\mathcal{X} \subseteq \mathcal{B}(r, r)$ for some r

Properties of $\mathcal{B}(r, s)$

Theorem

$\mathcal{B}(r, r)$ is the set of permutations sortable by $r - 1$ copies of the male-female sorting machine in series.



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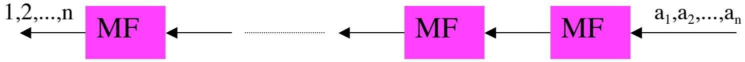
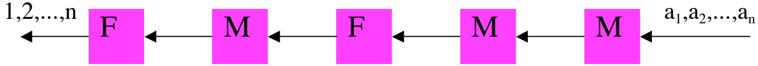


Theorem

$\mathcal{B}(r, s)$ is the set of permutations sortable by $r - 1$ male and $s - 1$ female sorting machines in any prescribed order in series.



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Let b_n be the number of permutations of length n in $\mathcal{B}(r, s)$. Then

$$b_n = rsb_{n-1} - 2!\binom{r}{2}\binom{s}{2}b_{n-2} + 3!\binom{r}{3}\binom{s}{3}b_{n-3} - \cdots$$

Main theorem

Theorem

Let $\mathcal{X}$ be any finitely based strong sorting class and let t_n be the number of permutations in $\mathcal{X}$ of length n. Then

$$\sum_{n=0}^{\infty} t_n x^n$$

is a rational function.

- D. E. Knuth: *Fundamental Algorithms, The Art of Computer Programming* Vol. 1 (Second Edition), Addison-Wesley, Reading, Mass. (1973).
- T. Mansour, A. Vainshtein: Avoiding maximal parabolic subgroups of S_k, Discrete Mathematics and Theoretical Computer Science 4 (2000), 67–77.