Permutation classes of polynomial growth

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Outline of talk

1. Background and previous work
2. Deciding polynomial growth
3. Enumerating polynomial growth classes
4. A hint at the proofs

Terminology

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- Generating function of $\mathcal{X}$

$$f(u) = \sum_{n=0}^{\infty} |\mathcal{X}_n| u^n$$

History – I

Theorem (Erdős-Szekeres, 1935)

A pattern class is finite if and only if its basis contains an increasing permutation and a decreasing permutation.

"$\mathrm{Av}(12 \cdots r, s \cdots 21)$ is finite."

History – II

Theorem (Marcus-Tardős, 2004)

If a pattern class $\mathcal{X}$ does not contain every permutation then, for some constant c, and all n

$$|\mathcal{X}_n| \leq c^n$$

"$\mathrm{Av}(B)$ is exponentially bounded if B is non-empty."

History – III

Theorem (Kaiser-Klazar, 2003)

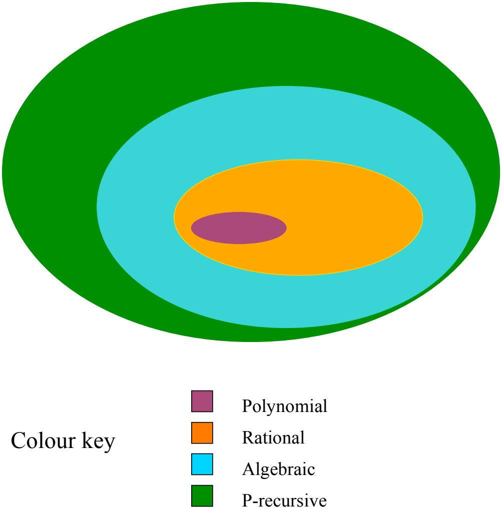
If a pattern class $\mathcal{X}$ has

$$|\mathcal{X}_n| < Fib_n \text{ for some } n$$

then $|\mathcal{X}_n|$ is a polynomial for all sufficiently large n

"If the growth rate of a class is less than τ^n ($\tau = \frac{1+\sqrt{5}}{2}$) the class has polynomial growth."

Landscape of classes by enumerative properties



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- Huczynska-Vatter:
  - Reproved KK's results and characterised polynomial growth classes in terms of "grid classes" of matchings.
  - It is decidable from the basis B whether $\mathrm{Av}(B)$ has polynomial growth

The decision problem - I

Theorem (H-V, and implicit in K-K)

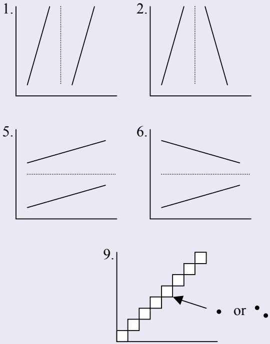
$\mathrm{Av}(B)$ *has polynomial growth if and only if it does not contain arbitrary long permutations of any of the forms*

1. $21436587\cdots$,
2. *its reverse,*
3. $a_1 b_1 a_2 b_2 \cdots$ *with* $\{a_1, a_2 \ldots\} < \{b_1, b_2 \ldots\}$
4. *its inverse*

The decision problem - II

Theorem (Different approach based on Ramsey theory)

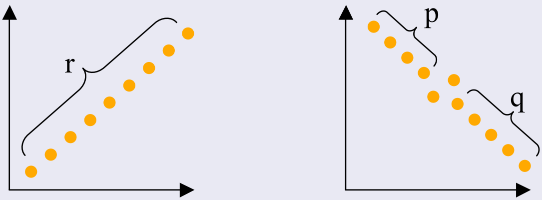
$\mathrm{Av}(B)$ *has polynomial growth if and only if* B *contains a permutation of each of the following shapes*



The decision problem - III

Corollary (of last theorem)

If $|B| = 2$ *then* $\mathrm{Av}(B)$ *has (non-zero) polynomial growth if and only if (to within symmetry) the permutations of* B *look like*



The decision problem - IV

Corollary (of last theorem)

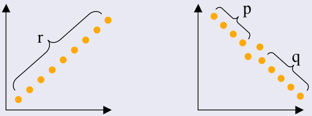
Let $\mathrm{Av}(\alpha, \beta, \gamma)$ *have polynomial growth. Then, up to symmetry and re-ordering* α, β, γ*, we have one of seven cases each pinning down the forms of* α, β, γ *(see abstract).*

For four or more restrictions the situation becomes too complicated to classify all the cases — and not particularly interesting to do so!

Enumeration with two restrictions - I

Theorem

If α, β have the form



then $\mathrm{Av}(\alpha, \beta)$ *is enumerated by a polynomial of degree d where*

$$(r-1)(p+q)-1 \leq d \leq \begin{cases} (r-1)^2(p+q)-r & \text{if } p>0 \text{ and } q>0, \\ (r-1)^2(p+q)-1 & \text{if } p=0 \text{ or } q=0. \end{cases}$$

Enumeration with two restrictions - II

Theorem

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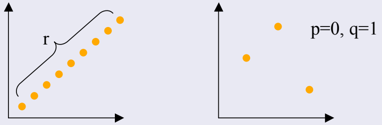

then $\mathrm{Av}(\alpha, \beta)$ is enumerated by a polynomial of degree $2r - 3$ and leading coefficient c_{r-3} (Catalan number)

Enumeration with two restrictions - III

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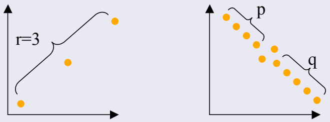

then $\mathrm{Av}(\alpha, \beta)$ is enumerated by a polynomial of degree $2p + 2q + 1$ (if $p, q > 0$) or $2p + 2q$ ($p = 0$ or $q = 0$)

A hint at the enumeration proofs

- Lower bounds — explicit exhibition of enough permutations in the class
- Upper bounds — several applications of Erdős-Szekeres

Irreducible permutations

Definition

A permutation is irreducible if it has no segment of the form $i + 1, i$.

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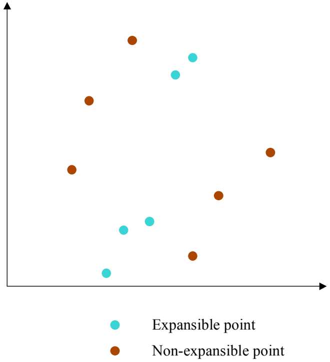
- If the irreducibles in a pattern class have maximal length m the class has polynomial growth of degree at most $m-1$ and possibly less.

Lower bounds

Produce irreducible permutations and large "expansible" subsequences



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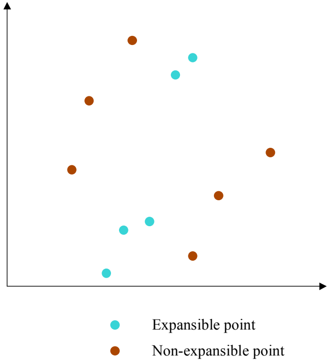
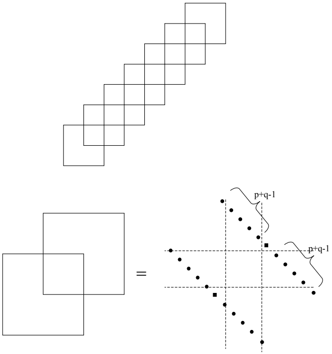
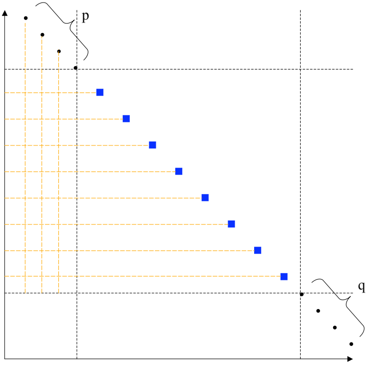




Upper bounds via longest decreasing subsequence

An irreducible permutation in $\mathrm{Av}(\alpha, \beta)$ and marked longest decreasing subsequence



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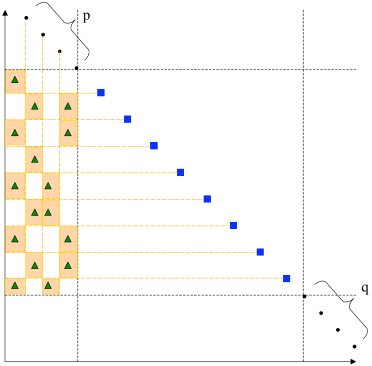
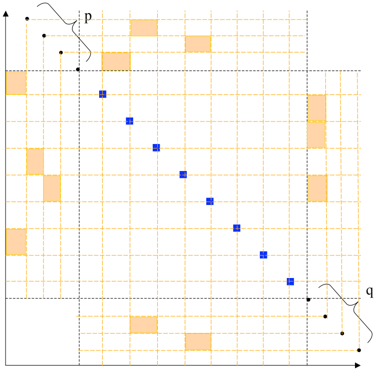
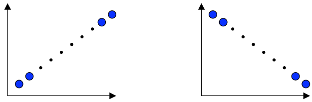


Upper bounds via longest decreasing subsequence

An irreducible permutation in $\mathrm{Av}(\alpha, \beta)$ and marked longest decreasing subsequence - a bounded number of separating boxes



Beyond Erdős-Szekeres

1.  FINITE

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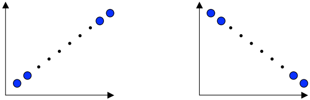
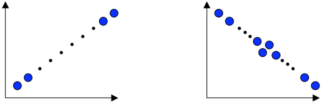
1.  FINITE
2.  POLYNOMIAL

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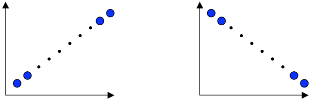
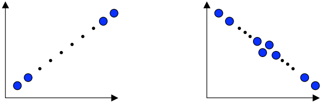
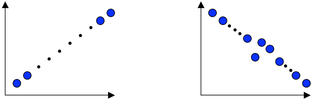
1.  FINITE
2.  POLYNOMIAL
3.  RATIONAL; g.r. $< 3 + 2\sqrt{2}$

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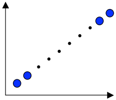
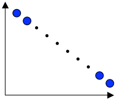
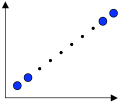
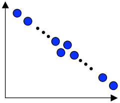
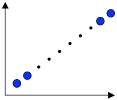
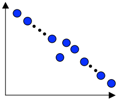
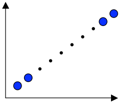
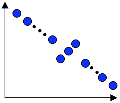
1.   FINITE
2.   POLYNOMIAL
3.   RATIONAL; g.r. $< 3 + 2\sqrt{2}$
4.   ????