Simple permutations, partial well-order, and enumeration

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Plan of talk

- Setting the scene

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- Algebraic generating functions

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- Simple permutations
- Finitely many restrictions
- Algebraic generating functions
- Some particular results

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- Closed sets are those of the form $\mathcal{S}[T]$ for some T

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In many cases solutions of these problems “go together”

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A permutation with no non-trivial component is called *simple*.

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Theorem. *The number of simple permutations of length* n *is asympotic to* $n!/e^2$.

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Elementary example

$$F = \{1, 12, 21\} \Longrightarrow \mathcal{F} = \mathcal{S}[3142, 2413]$$

$$A = \mathcal{S}[231]$$

has simple set F and so $A \subseteq \mathcal{F}$.

Wreath operations

Given a permutation α with $|\alpha| = n$ and permutations $\beta_1, \ldots, \beta_n$ define

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Example

$$\alpha = 231, \beta_1 = 21, \beta_2 = 132, \beta_3 = 321$$

$$\alpha(\beta_1, \beta_2, \beta_3) = 54 \mid 687 \mid 321$$

Partial well-order

Lemma. *If F and $\mathcal{F}$ are as above then, for all $\alpha \in F$ and $\beta_1, \ldots, \beta_n \in \mathcal{F}$, we have $\alpha(\beta_1, \ldots, \beta_n) \in \mathcal{F}$.*

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Theorem. *If F is finite then $\mathcal{F}$ is partially well-ordered by pattern containment.*

Proof $\mathcal{F}$ is an algebra under the wreath operations. Use Higman's theorem.

A finite number of restrictions

Corollary. *Every closed class with just a finite set of simple permutations is defined by a* finite *number of restrictions.*

Proof Main step: prove that $\mathcal{F}$ itself is defined by a finite number of restrictions. Then appeal to partial well-order.

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Lemma. *One of the following holds:*

- $\sigma = \alpha\beta$ *where either* $\alpha < \beta$ *or* $\alpha > \beta$
- $\sigma = \xi_1 \cdots \xi_m$ *where each* ξ_i *is a maximal component and the pattern formed by the* ξ_i *is simple.*

Structure

Let $F = \{1, 12, 21, \phi_3, \phi_4, \dots\}$.
Every permutation in $\mathcal{F} \setminus \{1\}$ has one of the forms

$$\phi(\beta_1, \dots, \beta_m)$$

where $\phi \in F \setminus \{1\}$.
The representation is unique except for when $\phi = 12$ or $\phi = 21$.

Notation $\phi(B_1, \dots, B_n)$ the set of all $\phi(\beta_1, \dots, \beta_n)$ with $\beta_i \in B_i$

Structure Theorem

$F, \mathcal{F}$ as above
$\mathcal{F}_+$ (resp. $\mathcal{F}_-$) permutations in $\mathcal{F}$ with no $\alpha\beta$ decomposition where $\alpha < \beta$ (resp. $\alpha > \beta$).

- $\mathcal{F} = \{1\} \cup 12(\mathcal{F}_+, \mathcal{F}) \cup 21(\mathcal{F}_-, \mathcal{F}) \cup \mathcal{G}$

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Enumeration of $\mathcal{F}$

Equations satisfied by ordinary generating functions

- $f(x) = x + f_+(x)f(x) + f_-(x)f(x) + g(x)$
- $f_+(x) = x + f_-(x)f(x) + g(x)$
- $f_-(x) = x + f_+(x)f(x) + g(x)$
- $g(x) = \sum_{\phi \in F, |\phi| > 2} f(x)^{|\phi|}$

Generating function of $\mathcal{F}$

$f(x)$ satisfies

$$f^2 - f + (x+g)(f+1) = 0$$

In particular,

Theorem. *If $|F|$ is finite, $f(x)$ (the generating function of $\mathcal{F}$) is algebraic.*

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General idea

- Subclasses have the form $\mathcal{F}[T]$ for some finite number of restrictions T
- $\mathcal{F}$ is a disjoint union of terms of the general type $\phi(A_1, A_2, \ldots, A_n)$ where ϕ is simple.

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- $\mathcal{F}$ is a disjoint union of terms of the general type $\phi(A_1, A_2, \ldots, A_n)$ where ϕ is simple.

So, how do we enumerate $\phi(A_1, \ldots, A_n)[T]$?

Enumeration of $\phi(A_1, \cdots, A_n)[T]$

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- $\tau \preceq \alpha_1 \cdots \alpha_n$ if and only if τ has a block decomposition $\tau_1 \cdots \tau_l$ where each $\tau_i \preceq \alpha_{s(i)}$ for some $s(i)$ and the pattern of the τ_i is a subpermutation of ϕ.

Technical calculations omitted

- $\phi(A_1, \ldots, A_n)[T]$ can be expressed as a union and intersection of sets of the form $\phi(A_1[T_1], \ldots, A_n[T_n])$ where the restrictions T_j are subpermutations of the restrictions T.

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- $\phi(B_1, \ldots) \cap \phi(C_1, \ldots) = \phi(B_1 \cap C_1, \ldots, B_n \cap C_n)$
- Use inclusion-exclusion and argue by induction

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Theorem. *Every proper closed subclass of* $\mathcal{S}[231]$ *has a rational generating function.*

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Theorem. *If the generating function of F is algebraic then the generating function of $\mathcal{F}$ is algebraic.*