Sorting classes, the weak and strong orders

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Outline of talk

Permuting machines

- The output is a (non-deterministic) rearrangement of the input
- The names of the input items are immaterial; use names $1, 2, \ldots$
- If some input items are omitted the machine can rearrange the remaining ones as they were arranged in the original

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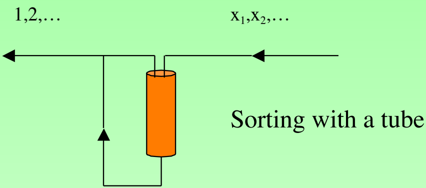
- Example: 312 $\preceq$ 7531462
- A *permutation class* is a set of permutations closed downwards in the $\preceq$ order
- The set of sortable inputs of a permuting machine is always a permutation class

Example



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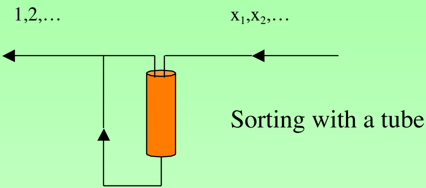

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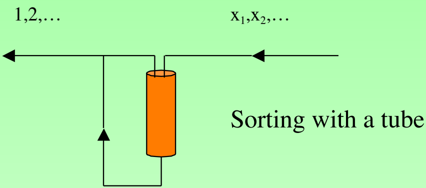

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- A permutation is tube-sortable if and only if involves neither 3241 or 4231 (i.e. $\{3241, 4231\}$ is the basis)
- If there are s_n sortable permutations of length n then

$$\sum_{n=0}^{\infty} s_n x^n = \frac{1}{2}(3 - x - \sqrt{1 - 6x + x^2})$$

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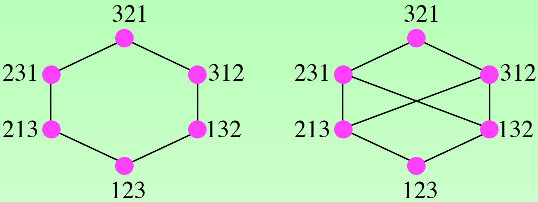
The weak and strong orders on S_3



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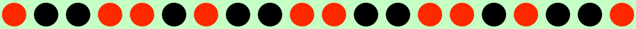
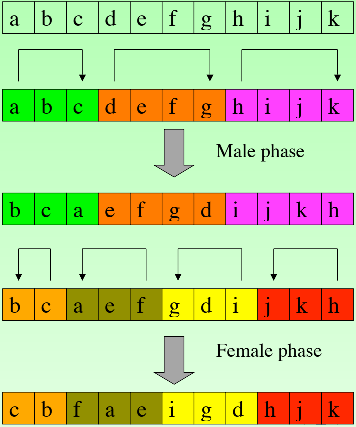


- Strong sorting class: permutation class closed downwards in the strong order
- Example: The male-female sorting machine

The male-female sorting machine

This machine operates in two phases: a *male* phase then a *female* phase.



Weak sorting classes

Weak sorting classes can be attacked because

Lemma

Let α, β be permutations. Then there exists γ such that

$$\alpha \leq_W \gamma \preceq \beta$$

if and only if there exists δ with

$$\alpha \preceq \delta \leq_W \beta$$

This not true for the strong order

Strong sorting classes

- The theory of strong sorting classes is quite different because the previous lemma does not hold for strong sorting classes.
- Example: $321 \preceq 3214 \leq_S 3412$ but no δ with $321 \leq_S \delta \preceq 3412$

The classes $\mathcal{C}(r)$

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$\mathcal{C}(r)$ is the class of all permutations which do not have a subsequence of $2r$ elements the first r being all larger than the last r.
This is a strong sorting class.

7 4 **12** 8 5 **9** 2 11 **6** 10 **1** **3**

Not in C(3)

The role of $\mathcal{C}(r)$

Theorem

If $\mathcal{X}$ is a strong sorting class not containing all permutations then $\mathcal{X} \subseteq \mathcal{C}(r)$ for some r

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Let c_n be the number of permutations of length n in $\mathcal{C}(r)$. Then

$$c_n = r^2 c_{n-1} - 2!\binom{r}{2}^2 c_{n-2} + 3!\binom{r}{3}^2 c_{n-3} - \cdots$$

Main theorem

Theorem

Let $\mathcal{X}$ be any finitely based strong sorting class and let t_n be the number of permutations in $\mathcal{X}$ of length n. Then

$$\sum_{n=0}^{\infty} t_n x^n$$

is a rational function.