Some basic math facts and strategies
COSC242: Algorithms and Data Structures

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1 Basic Algebra

In the following there is no special meaning to different characters. All can be treated as mathematical variables.

1.1 Distributive laws

$$
\begin{aligned}
&a(b+c) = ab + ac \\
&(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd \\
&a(b+c+d) = ab + ac + ad \\
&(a+b)(c+d+e) = a(c+d+e) + b(c+d+e) = ac + ad + ae + bc + bd + be \\
&(a+b)^2 = a^2 + 2ab + b^2 \\
&(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}
$$

1.2 Division laws

$$
\begin{aligned}
&\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \\
&\frac{a}{b} + c = \frac{a+bc}{b} \\
&\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \\
&\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b}\frac{d}{c}
\end{aligned}
$$

1.3 Exponent laws

$a^x a^y = a^{x+y}$
$\frac{a^x}{a^y} = a^{x-y}$
$a^{x+1} = a.a^x$
$2^x + 2^x = 2.2^x = 2^{x+1}$

1.4 Logarithm laws

We assume all logs are to base 2. The symbol $\implies$ should be read as "implies".

$\log 1 = 0$
$\log 2 = 1$
$x = 2^n \implies \log x = n$
$\log(ab) = \log a + \log b$
$\log 2^x = x$
$\log a^x = x \log a$

1.5 Summation laws

$$\sum_1^N x = 1 + 2 + 3 + \cdots + N$$

pair first and last term, second and second last, third and third last etc
$= (1 + N) + (2 + N - 1) + (3 + N - 2) + \cdots$
each of the terms in brackets equals $(1 + N)$
$= (1 + N) + (1 + N) + (1 + N) + \cdots$
there are N/2 such terms
$= (N + 1)N/2$

Actually, if N is even there are $N/2$ such terms. If N is odd, the middle term is left unpaired, but is equal to $(N + 1)/2$. In either case, the outcome is the same.

1.6 Inequality laws

$a \leq b \implies a - b \leq 0$
$a \leq b \implies a + c \leq b + c$
$a \leq b \implies a - c \leq b$ for $c \geq 0$
$a \leq b \implies a \leq b + c$ for $c \geq 0$
$a \leq b \implies -a \geq -b$
$a + b \leq c \implies a \leq c$ for $b \geq 0$

2 Simple proof strategies

2.1 Equalities

When we're trying to prove equalities, we want to make the left hand side of an equation equal to the right hand side. Some general things to try:

- expand out all brackets using the relevant distributive laws
- gather (or factorise) parts of the equation by reversing the distributive laws
- for fractions, put everything over the same denominator
- for fractions, multiply (or divide) the numerator and denominator by the same factor
- subtract equal quantities from the LHS and RHS
- add and subtract the same quantity from one side so you can get it in the same form as the other side.

2.2 Inequalities

All the strategies for equalities can be used for inequalities too. There is one extra thing you can do. Say you are trying to prove that:

$a \leq b$

Then if you can add positive terms to the LHS and show equality, the original inequality must be true. Essentially:

$a + x \leq b \implies a \leq b$ if $x \geq 0$.

So it is OK, in your proof, to add positive terms to the LHS as long as this results in an equality down the track. This is a bit counter-intuitive because it feels like we are doing this:

$a \leq b \implies a + x \leq b$ for $x \geq 0$.

But we are definitely *not* doing that.

Similarly, you can always multiply the LHS by a positive constant:

$ax \leq b \implies a \leq b$ if $x > 0$.