COSC342 Tutorial

Basic Mathematics

(Vectors and Matrices)

1 Notation

We will represent vectors as bold lower-case symbols, $\mathbf{v}$. You may also see vectors represented with a bar or an arrow over the symbol, $\bar{v}$ or $\overrightarrow{v}$, or a bar or tilde under the symbol, $\underline{v}$ or $\underset{\sim}{v}$. The components of a vector can be represented either as a list in round brackets, $\mathbf{v} = (x, y, z)$, or as a matrix with one column,

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since the one-column-matrix form takes up a lot of space, we'll sometimes write that as a transposed matrix, $\mathbf{v} = \begin{bmatrix} x & y & z \end{bmatrix}^T$. Matrices will generally be represented by uppercase letters, M, and scalars (plain numbers) as lower case letters in italics, a.

For scalar numbers we normally interpret $a \cdot b$, $a \times b$, and ab as all meaning the product of a and b. For vectors, $\cdot$ and $\times$ have specific and different meanings, and we can also multiply vectors by scalars. To avoid confusion, we will always write the product of a scalar and a vector as $a\mathbf{v}$, and never as $a \cdot \mathbf{v}$ or $a \times \mathbf{v}$. Similarly, the product of a scalar and a matrix is written aM, and the product of a matrix and a vector as M$\mathbf{v}$. When scalars are represented as symbols, we'll prefer the notation ab. However, when actual numbers are involved there is no risk of confusion, so it is OK to write $2 \times 3 = 6$ etc.

2 Vectors

2.1 Arithmetic

Let $\mathbf{u} = (1, -3, 2)$ and $\mathbf{v} = (3, 2, 0)$. Evaluate the following:

1. $2\mathbf{u}$
2. $\mathbf{u} + \mathbf{v}$
3. $\mathbf{u} - \mathbf{v}$
4. $2\mathbf{u} - 3\mathbf{v}$

2.2 Dot Products

Evaluate the following dot products:

1. $(1, 2, -2) \cdot (1, 2, -2)$
2. $(1, 2, -2) \cdot (-2, 2, 1)$
3. $(1, 2, -2) \cdot (2, 3, 1)$

What do these results tell you about each pair of vectors?

2.3 Cross Products

Evaluate the following cross products:

1. $(1, 2, -2) \times (-2, 2, 1)$
2. $(1, 2, -2) \times (-2, -4, 4)$
3. $(1, 2, -2) \cdot ((1, 2, -2) \times (-2, 2, 1))$

What does the answer to the second example tell you about the two vectors?

You can predict the answer to the third example without doing any calculation – why is that?

The brackets around the cross product in the third example aren't actually necessary – why not?

2.4 Vector Length

Calculate the length of the following vectors:

1. $(1, 1, 1)$
2. $(0, 4, 3)$
3. $(1, 2, -2)$

3 Points and Lines

1. Find the distance between the points $\mathbf{p}_0 = (1, 2, -2)$ and $\mathbf{p}_1 = (3, -2, 2)$.
2. Determine the parametric equation for the line that goes through the points $\mathbf{p}_0$ and $\mathbf{p}_1$.

4 Matrices

4.1 Arithmetic

Evaluate the following:

1. $2\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}3 & 2\\ 4 & 1\end{bmatrix}$

2. $\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & -2\\ -4 & 1\end{bmatrix}$

3. $\begin{bmatrix}1 & -2\\ -4 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}$

How can you reconcile the results of (2) and (3) with the fact that matrix multiplication is not commutative – that is that for matrices $\mathrm{AB} \neq \mathrm{BA}$?

4.2 Matrix Inverses

Finding the inverse of a matrix in general is rather complicated, and can often be avoided. For 2×2 matrices, however, there is a relatively simple formula. If

$$\mathrm{M} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

then

$$\mathrm{M}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

.

Let $\mathrm{A} = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix}3 & -1\\ -4 & 2\end{bmatrix}$. Compute the following:

1. B^{-1} and B^{-1}

2. $(\mathrm{AB})^{-1}$

How is $(\mathrm{AB})^{-1}$ related to A^{-1} and B^{-1}?

5 Matrices and Vectors

These following exercises make use of the matrix $\mathrm{A} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\end{bmatrix}$, and the vectors $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (-2, 2, 1)$.

5.1 Arithmetic

Compute the following:

1. $A\mathbf{u}$
2. $\mathbf{u}\mathbf{u}^T$
3. $\mathbf{u}^T\mathbf{v}$

The third example should be equal to the dot product of **u** and **v**, which you computed in Section 2.2, Exercise 3. Show that this is true for any pair of vectors with the same length.

5.2 Cross Products in Matrix Form

Given a 3-vector $\mathbf{w} = \begin{bmatrix} x & y & z \end{bmatrix}^T$, we can form a matrix

$$[\mathbf{w}]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

1. What is $[\mathbf{u}]_\times$
2. Compute $[\mathbf{u}]_\times \mathbf{v}$
3. This should be the same as the cross product, $\mathbf{u} \times \mathbf{v}$, which you computed in Section 2.3, Exercise 1. Show that this is true for any pair of 3-vectors.