COSC342 Tutorial

2D Transforms

(Composite Transformations, Shearing and Kinematic Chains Solutions)

1 Composite Transformations

We can create composite transformations by concatenating transformation matrices. This will result in a combined transformation matrix.

1. Explain the following: Matrix multiplication is associative, not commutative.

   Matrix multiplication is not commutative: the order is important. But matrix multiplication is associative, so we can calculate from right to left or left to right: $ABCD = (((AB)C)D) = (A(B(CD)))$.

2. Pre-multiplication defines the order how transform matrices are multiplied. It means that the first transformation you want to perform will be at the far right. Describe the order of performing a translation T_1 first and a rotation R_2 afterwards using the pre-multiplication definition.

   $M = R_2T_1$.

1.1 Scale and Translation

A translation about dx along the x-axis and dy along the y-axis is defined as:

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

A transformation matrix describing **scale** using a scale factor s is defined as:

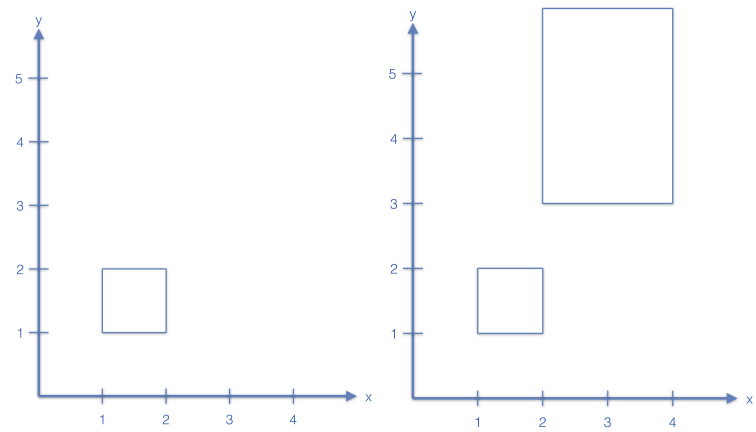
$$\mathrm{S} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. What is a scale matrix applying scale only along the x-axis?

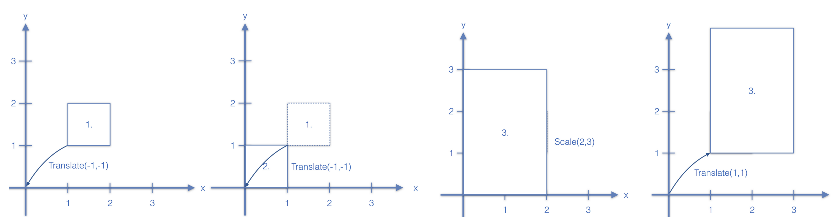
$$\mathrm{S_x} = \begin{bmatrix} sx & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Draw the result of the applying S_1 to the following figure.

$$\mathrm{S}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3. How can we scale an object without moving its origin?



4. Create a transformation matrix for the previous scaling operation.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Rotation

A rotation by an angle θ around the origin is defined by:

$$\mathrm{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. What are the required transformation steps to create a rotation around a point (x_0, y_0)

(a) Translate the coordinates so that the origin is at (x_0, y_0)
(b) Rotate by θ
(c) Translate back

2. How would a transformation matrix for a rotation around a point (x_0, y_0) would be created and what is the final transformation matrix?

$$\begin{bmatrix}1 & 0 & x_0\\ 0 & 1 & y_0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -x_0\\ 0 & 1 & -y_0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & x_0(1-\cos\theta)+y_0\sin(\theta)\\ \sin\theta & \cos\theta & y_0(1-\cos\theta)-x_0\sin\theta\\ 0 & 0 & 1\end{bmatrix}$$

3. If you apply the transformations in reverse order would we end up with the same result?

No, we would rotate around $(-x_0, -y_0)$:

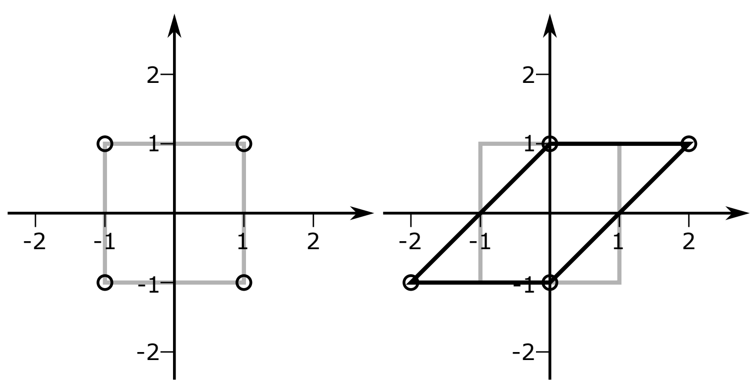
$$\begin{bmatrix}1 & 0 & -x_0\\ 0 & 1 & -y_0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & x_0\\ 0 & 1 & y_0\\ 0 & 0 & 1\end{bmatrix}$$

1.3 Shearing

We have seen three types of basic transform – translation, scaling, and rotation. Another type of transform is a *shear*. A horizontal shear can be implemented with the following transformation matrix:

$$\mathrm{S}_x = \begin{bmatrix}1 & m & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

1. What is the result of applying a horizontal shear with $m = 1$ to the corners of the square in the following figure?



2. The equivalent transform for a shear in the vertical direction is

$$\mathrm{S}_y = \begin{bmatrix}1 & 0 & 0\\ n & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

If you apply a vertical then a horizontal shear, is that the same as applying the same horizontal shear followed by the vertical shear?

No. If we apply the horizontal transform first, the combined transform is

$$\begin{bmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m & 0 \\ n & 1+mn & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

while if we apply the vertical transform first we get

$$\begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+mn & m & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 Kinematics (extension)

Suppose we have a model of a character's arm. The shoulder is at the origin, $(0,0)$, and when it is at rest the elbow is at $(0,-2)$, the wrist at $(0,-4)$, and the fingertips at $(0,-5)$, as shown on the left of Figure 1. We want to raise the arm so that it is raised, as shown on the right of Figure 1. In this pose, the elbow is at $(2,0)$, the wrist at $(2,2)$, and the fingertips at $(2,3)$.

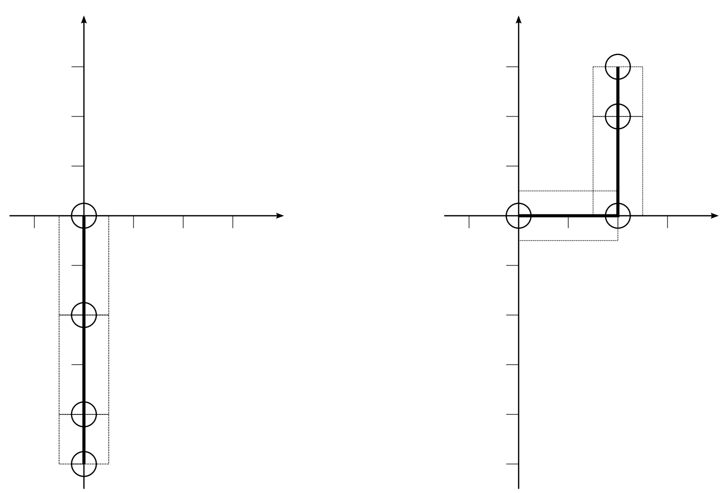


Figure 1: A simple arm at rest (left) and raised (right)

1. A simple way to animate the arm is to interpolate the co-ordinates. If a point moves from (x,y) at time $t=0$ to (x',y') at time $t=1$ then a linear interpolation at time t gives it's position as

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = (1-t)\begin{bmatrix} x \\ y \end{bmatrix} + t\begin{bmatrix} x' \\ y' \end{bmatrix}.$$

What are the positions of the joints at $t = 0.5$ in this model?

The shoulder stays at $(0, 0)$, all the other points end up at $(1, -1)$.

A better way to animate the arm is with a *kinematic chain*. A simple kinematic chain consists of $k+1$ points, $\mathbf{p}_0, \mathbf{p}_1, \ldots \mathbf{p}_k$. The first point, $\mathbf{p}_0$, is at the origin, and subsequent points are defined by an angle and distance *relative to the previous point.* This is illustrated in Figure 2.

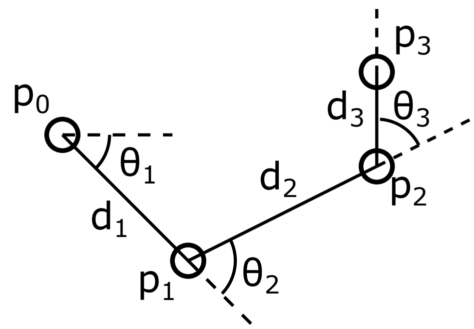


Figure 2: A simple kinematic chain.

The state of the kinematic chain is given by $\{\theta_1, d_1, \theta_2, d_2, \ldots, \theta_k, d_k\}$, where $\mathbf{p}_0$ is at the origin, d_i is the distance between points $\mathbf{p}_{i-1}$ and $\mathbf{p}_i$, and θ_i is the angle between the lines which meet at $\mathbf{p}_i$.

To figure out the co-ordinates of the points, we can express the distances between the points as translation matrices, and the angles as rotation matrices:

$$\mathrm{T}_i = \begin{bmatrix} 1 & 0 & d_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The position of the point $\mathbf{p}_i$ is then given by

$$\mathbf{p}_i = \mathrm{R}_1 \mathrm{T}_1 \mathrm{R}_2 \mathrm{T}_2 \ldots \mathrm{R}_i \mathrm{T}_i \mathbf{p_0}.$$

Note that the transforms are applied backwards from the end of the chain. This is because rotating the elbow (for example) moves the forearm, wrist, and hand as a single unit.

A kinematic chain defining the arm we used above has four points, so $k = 3$. The distances between the joints are $d_1 = 2$, $d_2 = 2$, and $d_3 = 1$.

2. What are the positions of the joints when all of the angles are zero?

$$\mathbf{p}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\mathsf{T} \quad \mathbf{p}_1 = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^\mathsf{T} \quad \mathbf{p}_2 = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^\mathsf{T} \quad \mathbf{p}_3 = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}^\mathsf{T}$$

3. What are the angles for the 'rest' position when the arm is lying along the negative y-axis, as illustrated on the left of Figure 1?

$$\theta_1 = -90^\circ \quad \theta_2 = 0^\circ \quad \theta_3 = 0^\circ$$

4. What are the angles when the arm is raised, as illustrated on the right of Figure 1?

$$\theta_1 = 0^\circ \quad \theta_2 = 90^\circ \quad \theta_3 = 0^\circ$$

5. We can now animate the arm by interpolating the angles rather than the positions of the points. What are the angles half-way between the two poses?

$$\theta_1 = -45^\circ \quad \theta_2 = 45^\circ \quad \theta_3 = 0^\circ$$

6. How would you find where the fingertips are halfway between the two poses? I'll write $\mathrm{R}(a)$ for a rotation by a degrees, and $\mathrm{T}(b)$ for a translation of b units along the x-axis.

$$\begin{aligned}
\mathbf{p}_3 =& \mathrm{R}_1\mathrm{T}_1\mathrm{R}_2\mathrm{T}_2\mathrm{R}_3\mathrm{T}_3\mathbf{p}_0\\
=& \mathrm{R}(-45)\mathrm{T}(2)\mathrm{R}(45)\mathrm{T}(2)\mathrm{R}(0)\mathrm{T}(1)\mathbf{p}_0\\
=& \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\\
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=& \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} + 2 \\ \frac{3}{\sqrt{2}} \\ 1 \end{bmatrix}\\
=& \begin{bmatrix} \frac{3}{2} + \sqrt{2} + \frac{3}{2} \\ -\frac{3}{2} - \sqrt{2} + \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 + \sqrt{2} \\ -\sqrt{2} \\ 1 \end{bmatrix}
\end{aligned}$$

Note: Once you've figured out what matrices to apply and in what order, get a computer to do the maths!