Department of Computer Science,
University of Otago

Technical Report OUCS-2006-09

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Authors:

M.H. Albert
Department of Computer Science, University of Otago

Micah Coleman, Ryan Flynn
Department of Mathematics, University of Florida

Imre Leader
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge

Status: Submitted to *Annals of Combinatorics*

Department of Computer Science,
University of Otago, PO Box 56, Dunedin, Otago, New Zealand

http://www.cs.otago.ac.nz/research/techreports.html

PERMUTATIONS CONTAINING MANY PATTERNS

M. H. ALBERT, MICAH COLEMAN, RYAN FLYNN, AND IMRE LEADER

ABSTRACT. It is shown that the maximum number of patterns that can occur in a permutation of length n is asymptotically 2^n. This significantly improves a previous result of Coleman.

1. Introduction

Given a sequence $\mathbf{t} = t_1, t_2, \ldots, t_k$ of distinct elements from some totally ordered set, there is a unique permutation τ of $[k] = \{1, 2, \ldots, k\}$ with the property that for all $1 \leq i, j \leq k$, $t_i < t_j$ if and only if $\tau(i) < \tau(j)$. We call τ the *pattern* of $\mathbf{t}$. For example, the pattern of 5, 10, 2 written in one line notation is 231. In other words, the sequence representing τ is obtained from $\mathbf{t}$ simply by replacing each element of $\mathbf{t}$ by its rank in $\mathbf{t}$.

Let σ be a permutation of length n, written in one-line notation as $\sigma(1)\sigma(2)\cdots\sigma(n)$, and thought of as a sequence of length n. For each non-empty subset X of $[n]$ define σ_X to be the pattern of that subsequence of σ whose indices belong to X. Define:

$$P(\sigma) = \{\sigma_X \,:\, \emptyset \neq X \subseteq [n]\}.$$

That is, $P(\sigma)$ is the set of patterns that occur in σ. Also define $h(n)$ to be the maximum value of $|P(\sigma)|$ taken over all permutations σ of length n.

Trivially, $h(n) \leq 2^n - 1$. Slightly more precisely, for any permutation σ of length n:

$$|P(\sigma)| \leq \sum_{k=1}^{n} \min\left(k!, \binom{n}{k}\right) \tag{1}$$

since not more than $k!$ patterns of length k can occur. However, the expression on the right hand side of this inequality is easily seen to be asymptotically 2^n. At the 2003 conference on Permutation Patterns, Herb Wilf raised the issue of determining the (asymptotic) behaviour of $h(n)$, and exhibited a sequence of permutations which established that $h(n)$ exceeded the n^{th} Fibonacci number. Micah Coleman then demonstrated in [1] a sequence of permutations π_n, for n a perfect square,[1] for which:

$$|P(\pi_n)| > 2^{n-2\sqrt{n}+1}.$$

Of course this establishes that $h(n)^{1/n} \to 2$ (for all n, not just perfect squares, using the fact that $h(n)$ is non decreasing). However, this left open the question of whether or not $h(n)/2^n$ tends to 1 as n tends to infinity.

In this paper, we refine the counting arguments concerning the number of patterns in π_n, for n an even perfect square, and then extend the construction to all other values of n, in order to show that $|P(\pi_n)|/2^n \to 1$. Indeed, we will obtain:

$$h(n) > 2^n \left(1 - 6\sqrt{n}\, 2^{-\sqrt{n}/2}\right)$$

for all positive integers n.

2. The main construction

Let k be a positive integer and let $n = 4k^2$. Let s be the sequence:

$$s = (2k)\,(4k)\,(6k)\,\cdots\,(4k^2)$$

and consider the permutation π_n which in one line notation is defined by:

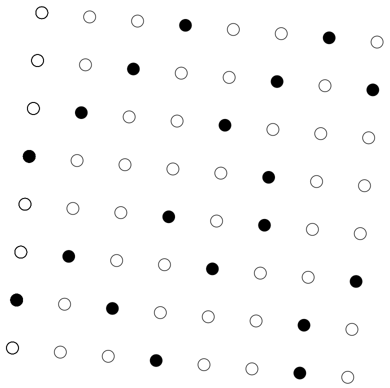
$$\pi_n = s\,(s-1)\,(s-2)\,\cdots\,(s-2k+1).$$

Here $s - i$ indicates the sequence obtained by subtracting i from each element of s. Generally, we will suppress the subscript on π_n when there is no risk of confusion. Informally, the graph of π is obtained by taking a standard orthogonal $2k \times 2k$ grid and rotating it slightly in the clockwise direction around its lower left hand corner. We associate to each subset X of (the indices of) π a $2k \times 2k$ 0-1 matrix, M_X, whose 1 entries correspond to the elements of the subset. We also view M_X as being partitioned into four $k \times k$ submatrices (called the *corner submatrices*) in the usual way, that is, so that they form a 2×2 block decomposition of M_X. We say that X (or M_X) is *ample* if each $k \times k$ corner submatrix of M_X has no zero rows or zero columns. An example is shown in Figure 1.

Proposition 1. *The number of ample matrices is greater than*

$$2^n \left(1 - \frac{4\sqrt{n}}{2^{\sqrt{n}/2}}\right).$$

[1] We have adjusted the notation slightly from that of [1] — what was there called π_k we are calling π_{k^2} so that the subscript is equal to the length of the permutation.



$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

FIGURE 1. The graph of the permutation π_{64}, an ample subset of its elements indicated by filled circles, together with the corresponding matrix divided into its corner submatrices.

Proof. Recall that $n = 4k^2$. Suppose that we sample an $n \times n$ 0-1 matrix uniformly at random from among all $n \times n$ 0-1 matrices. The probability that any particular row or column sum of one of the corner submatrices is 0 is $1/2^k$. There are $8k$ such sums which must all be non zero in order for the matrix to be ample. However, the probability that any of them are 0 is less than $8k/2^k$. So, the probability that all are non zero is greater than

$$1 - \frac{8k}{2^k} = 1 - \frac{4\sqrt{n}}{2^{\sqrt{n}/2}},$$

which is equivalent to the stated result. □

Proposition 2. *Let X and Y be ample sets. Then $\pi_X = \pi_Y$ implies $X = Y$.*

Proof. We must show that, if X is ample, then it can be reconstructed from just the permutation π_X. Since X is ample, the column sum of both the top half and bottom half of each column of M_X is non zero. Therefore, there are $2k - 1$ descents in π_X, corresponding to the transitions between columns of M_X. Thus, we can associate the elements of π_X with their correct columns. However, this argument applies equally well to the rows of M_X — as is most easily seen by considering π^{-1}. Determining the row and column that represents each element of π_X is exactly the same as reconstructing X. □

Combining these two results we have:

Theorem 3. *If n is an even perfect square, then*

$$h(n) > 2^n \left(1 - \frac{4\sqrt{n}}{2^{\sqrt{n}/2}}\right).$$

We will refer to the second term inside the parentheses above as the *correction term* for this estimate.

3. Refinements

It is easy to extend the above arguments to give lower bounds on $h(n)$ that are valid for *all* values of n. We can do this by using the basic construction of the previous section, and adding some extra elements in appropriate places to construct permutations π_n of length n that contain many patterns.

First suppose that $n = 4k^2 + l$ where $0 < l < 2k$. Take the grid associated to the permutation π_{4k^2} and add a (partial) column on the right hand side at the bottom containing not more than k elements, and, if necessary, a partial row on top at the right hand side, also not containing more than k elements, so that the total number of elements added is l. As before, rotate this grid slightly, and view the result as the graph of a permutation, π_n. An example is shown in Figure 2. Call the elements of this permutation arising from the original grid defining π_{4k^2} the *main* elements, and the remaining elements the *extra* elements. Define a subset of the indices of π_n to be *ample* if its intersection with the main elements would be ample for π_{4k^2}.

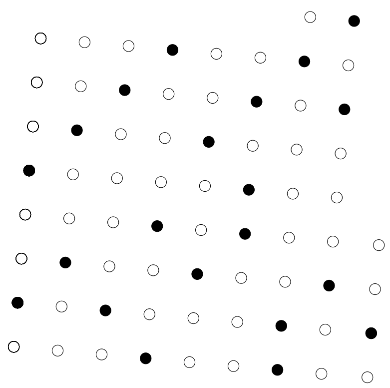


Figure 2. The graph of the permutation π_{70}, together with the matrix associated with a particular ample subset of its elements indicated by filled circles.

Proposition 4. *Let X and Y be ample sets. Then $\pi_X = \pi_Y$ implies $X = Y$.*

Proof. As before, we must describe how to reconstruct X from π_X. However, we can identify the extra elements (and hence the main elements) in π_X. If there are any belonging to the new partial column, then they are exactly the elements following the $(2k)^{\text{th}}$ descent, while those belonging to the new partial row, if such exist, are exactly those lying above the maximum element of the first k columns. Since the main elements form an ample subset of π_{4k^2} we can use the preceding result to identify their values. Once the values of the main elements are known, so are the values of the extra elements. □

Therefore, for such n,

$$h(n) \geq |P(\pi_n)| > 2^{4k^2} \left(1 - \frac{8k}{2^k}\right) 2^l.$$

Certainly $k \leq \sqrt{n}/2$, but also $(2k + 1/2)^2 > n$ so $k > (\sqrt{n} - 1/2)/2$. Applying these estimates we obtain:

$$h(n) > 2^n \left(1 - \frac{2^{9/4}\sqrt{n}}{2^{\sqrt{n}/2}}\right).$$

This differs from our previous estimate by a factor of $2^{1/4}$ in the correction term.

For $n = 4k^2 + 2k$, we switch to a grid consisting of $2k + 1$ columns of size $2k$ and define π_n appropriately. As in the previous section, we define the four corner submatrices, except now those on the right hand side of the matrix are $k \times (k + 1)$ instead of $k \times k$. The probability of a subset of the matrix not being ample is not as much as:

$$\frac{2(2k + 1)}{2^k} + \frac{2k}{2^k} + \frac{2k}{2^{k+1}} = \frac{7k + 2}{2^k}.$$

Using the same bounds as before (which still apply) plus trivial estimates for $k \leq 2$ it is easy to check that the bound

$$h(n) > 2^n \left(1 - \frac{2^{9/4}\sqrt{n}}{2^{\sqrt{n}/2}}\right)$$

still applies in this case. We can proceed from this point with the half-row/half-column construction again (possibly at a penalty of another factor of $2^{1/4}$ in the correction term) as far as $n = (2k + 1)^2$. At this point we pause for a detailed re-evaluation. In a $(2k+1)\times(2k+1)$ grid, divided into corner submatrices of sizes $k \times k$, $k \times (k + 1)$, $(k + 1) \times k$

and $(k+1) \times (k+1)$, the probability that a subset is not ample is less than:

$$\frac{2k}{2^k} + 2\left(\frac{k}{2^{k+1}} + \frac{k+1}{2^k}\right) + \frac{2(k+1)}{2^{k+1}} = \frac{6k+3}{2^k}.$$

Since $k = (\sqrt{n} - 1)/2$, this equals

$$\frac{(3\sqrt{2})\sqrt{n}}{2^{\sqrt{n}/2}}.$$

We can pursue these constructions through to the next even perfect square, and, allowing for a further penalty of $\sqrt{2}$ in the correction term (which we leave to the reader to verify is generous), obtain:

Theorem 5. *For all positive integers n,*

$$h(n) > 2^n \left(1 - \frac{6\sqrt{n}}{2^{\sqrt{n}/2}}\right).$$

4. Conclusions

It would be interesting to know just how close to 2^n the value of $h(n)$ actually is. A more careful analysis of the various steps in moving from one square grid to the next might well provide a small improvement in the constant factor of the correction term of our estimate. Similarly, an analysis of conditions weaker than ample which none the less would allow for a reconstruction result might actually improve the asymptotic form of the correction term. However, the simplicity of the main construction (for $n = 4k^2$) and of the proof that ample subsets can be reconstructed from their patterns, together with the lack of any great *need* for more precise estimates of $h(n)$ somewhat dampens our enthusiasm for further investigations in that direction. Of perhaps greater interest would be to investigate the distribution of the statistic $|P(\pi)|$ as π ranges over permutations of length n.

We would like to thank Herb Wilf for having posed such an interesting problem!

References

[1] Micah Coleman. An answer to a question by Wilf on packing distinct patterns in a permutation. *Electron. J. Combin.*, 11(1):Note 8, 4 pp. (electronic), 2004.

Department of Computer Science, University of Otago
E-mail address: malbert@cs.otago.ac.nz

Department of Mathematics, University of Florida

Department of Mathematics, University of Florida

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge