Longest Pattern-Avoiding Subsequences of Random Permutations

M. H. Albert (Otago)

PP 2006, Reykjavik

A story . . .

A long time ago on an island far far away . . .

A story . . .

A long time ago on an island far far away . . .

Herb Wilf talked about the Baik, Deift, Johansson result on the distribution of the length of the longest increasing subsequence in a random permutation and asked:

A story . . .

A long time ago on an island far far away . . .

Herb Wilf talked about the Baik, Deift, Johansson result on the distribution of the length of the longest increasing subsequence in a random permutation and asked:

> *What can be said about the distribution of the length of the longest increasing sequence in a permutation chosen at random from a pattern class $\mathcal{A}$?*

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.
- (Hammersley, 1972) That's true and (conjecture) $c = 2$.

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.
- (Hammersley, 1972) That's true and (conjecture) $c = 2$.
- (Various, 1977) That's true.

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.
- (Hammersley, 1972) That's true and (conjecture) $c = 2$.
- (Various, 1977) That's true.
- (Odlyzko and Rains, 1985+) Simulation (conjecture) the length is tightly concentrated around the mean (i.e. the variance is small).

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.
- (Hammersley, 1972) That's true and (conjecture) $c = 2$.
- (Various, 1977) That's true.
- (Odlyzko and Rains, 1985+) Simulation (conjecture) the length is tightly concentrated around the mean (i.e. the variance is small).
- (Frieze, Bollobas and Brightwell, 1991-92) That's true.

Longest increasing subsequences

- (Conjecture: Ulam, 1960) The length of the longest increasing subsequence of a random permutation from $\mathcal{S}_n$ is (asymptotically in expectation) $c\sqrt{n}$.
- (Hammersley, 1972) That's true and (conjecture) $c = 2$.
- (Various, 1977) That's true.
- (Odlyzko and Rains, 1985+) Simulation (conjecture) the length is tightly concentrated around the mean (i.e. the variance is small).
- (Frieze, Bollobas and Brightwell, 1991-92) That's true.
- (Baik, Deift and Johansson, 1999) We know everything about the distribution.

A diversion

The rest of the talk is not about increasing subsequences. So, if you want to keep thinking about them, try this:

A diversion

The rest of the talk is not about increasing subsequences. So, if you want to keep thinking about them, try this:

> *I have n cards numbered 1 through n and I've shuffled them well. I will deal them all out one at a time, and each time I deal a card you can choose to "accept" it* provided *that the cards you accept form an increasing sequence. Playing optimally (i.e. trying to accept as many cards as possible) how many cards do you expect to accept?*

The wrong end of the stick

Somehow I remembered Herb's question as:

The wrong end of the stick

Somehow I remembered Herb's question as:

What can be said about the distribution of the length of the longest subsequence belonging to a given pattern class $\mathcal{A}$ in a random permutation?

The wrong end of the stick

Somehow I remembered Herb's question as:

> *What can be said about the distribution of the length of the longest subsequence belonging to a given pattern class $\mathcal{A}$ in a random permutation?*

I'll tell the early (i.e. easy) parts of this story and, in the tradition of the area, add a conjecture of my own.

Notation

- Throughout, $\mathcal{A}$ is some proper, infinite pattern class (i.e. set of permutations closed under taking subpermutations).

Notation

- Throughout, $\mathcal{A}$ is some proper, infinite pattern class (i.e. set of permutations closed under taking subpermutations).
- The *growth rate*, or *Stanley-Wilf* limit of $\mathcal{A}$:

$$s_{\mathcal{A}} = \limsup_{n \to \infty} |\mathcal{A} \cap \mathcal{S}_n|^{1/n} .$$

Notation

- Throughout, $\mathcal{A}$ is some proper, infinite pattern class (i.e. set of permutations closed under taking subpermutations).
- The *growth rate*, or *Stanley-Wilf* limit of $\mathcal{A}$:

$$s_{\mathcal{A}} = \limsup_{n \to \infty} |\mathcal{A} \cap \mathcal{S}_n|^{1/n}.$$

- Π_n is a random variable uniformly distributed on $\mathcal{S}_n$. $L_{\mathcal{A}}(\Pi_n)$ is the random variable whose value is the length of the longest subsequence of (an observation of) Π_n whose pattern belongs to $\mathcal{A}$.

Preliminaries

Lemma

$$\mathbf{Pr}\left(L_{\mathcal{A}}(\Pi_n) \geq 2e\sqrt{s_{\mathcal{A}} n}\right) < e^{-2e\sqrt{s_{\mathcal{A}} n}}.$$

Preliminaries

Lemma

$$\mathbf{Pr}\left(L_{\mathcal{A}}(\Pi_n) \geq 2e\sqrt{s_{\mathcal{A}} n}\right) < e^{-2e\sqrt{s_{\mathcal{A}} n}}.$$

This is simply a matter of counting – in expectation fewer than $e^{-2e\sqrt{s_{\mathcal{A}} n}}$ subsequences of length $\lceil 2e\sqrt{s_{\mathcal{A}} n} \rceil$ of a permutation of length n can belong to $\mathcal{A}$. Therefore, this is an upper bound for the probability that one exists.

Expectation

Theorem

If $\mathcal{A}$ is sum or difference closed, then there is a constant $c_{\mathcal{A}}$ with $1 \leq c_{\mathcal{A}} \leq e^2 s_{\mathcal{A}}$ such that:

$$\lim_{n\to\infty} \frac{\mathbf{E}\left(L_{\mathcal{A}}(\Pi_n)\right)}{\sqrt{n}} = 2\sqrt{c_{\mathcal{A}}}.$$

The proof *is* (not just "essentially is") the same as Hammersley's for the class $\mathcal{I}$ of increasing permutations.

Concentration

Theorem

For $\alpha > 1/3$ *and* $\beta < \min(\alpha, 3\alpha - 1)$

$$\mathbf{Pr}\left(|L_{\mathcal{A}}(\Pi_n) - \mathbf{E}\, L_{\mathcal{A}}(\Pi_n)| \geq n^{\alpha}\right) < \exp(-n^{\beta}).$$

This time the proof is Frieze's (in fact he foreshadows the possibility of such extensions at the end of his paper).

Known values

Let $\mathcal{I}_k$ be the class of permutations avoiding $k(k-1)\cdots 321$.

$$c_{\mathcal{I}_k} = s_{\mathcal{I}_k} = (k-1)^2.$$

To boldly go . . .

Conjecture

For all $\mathcal{A}$, $c_{\mathcal{A}} = s_{\mathcal{A}}$.

Expectation + Concentration ⇒ Preservation

There are a number of different constructions that take pattern classes as input and produce pattern classes as output. A natural question to ask is:

How do these constructions affect the constants $c_\bullet$ *and* $s_\bullet$*?*

Expectation + Concentration ⇒ Preservation

There are a number of different constructions that take pattern classes as input and produce pattern classes as output. A natural question to ask is:

How do these constructions affect the constants $c_\bullet$ *and* $s_\bullet$*?*

Obviously our hope is that the constructions preserve positive instances of the conjecture!

Union

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be proper pattern classes and let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Then:

$$s_{\mathcal{C}} = \max(s_{\mathcal{A}}, s_{\mathcal{B}})$$
$$c_{\mathcal{C}} = \max(c_{\mathcal{A}}, c_{\mathcal{B}}).$$

Sum

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be proper pattern classes and let $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$. Then:

$$s_{\mathcal{C}} = \max(s_{\mathcal{A}}, s_{\mathcal{B}})$$
$$c_{\mathcal{C}} = \max(c_{\mathcal{A}}, c_{\mathcal{B}}).$$

Juxtaposition

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be proper pattern classes and let $\mathcal{C} = \mathcal{A}\mathcal{B}$. Then:

$$\begin{aligned} s_{\mathcal{C}} &= s_{\mathcal{A}} + s_{\mathcal{B}} \\ c_{\mathcal{C}} &= c_{\mathcal{A}} + c_{\mathcal{B}}. \end{aligned}$$

Merge

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be proper pattern classes whose intersection is finite and let $\mathcal{C} = \mathrm{Merge}(\mathcal{A}, \mathcal{B})$. Then:

$$\sqrt{s_{\mathcal{C}}} = \sqrt{s_{\mathcal{A}}} + \sqrt{s_{\mathcal{B}}}$$
$$\sqrt{c_{\mathcal{C}}} = \sqrt{c_{\mathcal{A}}} + \sqrt{c_{\mathcal{B}}}.$$

Removing, or at least weakening, the rather stringent condition here would be desirable.

Rotation

Proposition

Let $\mathcal{A}$ be a proper pattern class and let $\mathcal{B} = \mathrm{Rot}(\mathcal{A})$ (rotations of $\mathcal{A}$). Then $s_{\mathcal{B}} = s_{\mathcal{A}}$ and $c_{\mathcal{B}} = c_{\mathcal{A}}$.

Salted fish

No doubt some of the preceding results could be strengthened and other preservation results could be found.

Computing $c_{\mathcal{A}}$

Can we compute or estimate $c_{\mathcal{A}}$ *for some classes* $\mathcal{A}$ *not covered by the preservation theorems?*

Computing $c_{\mathcal{A}}$

Can we compute or estimate $c_{\mathcal{A}}$ for some classes $\mathcal{A}$ not covered by the preservation theorems?

- No real progress on “closed form” computation.

Computing $c_{\mathcal{A}}$

Can we compute or estimate $c_{\mathcal{A}}$ for some classes $\mathcal{A}$ not covered by the preservation theorems?

- No real progress on "closed form" computation.
- Estimation or experiment requires us to have available good algorithms for finding $\mathcal{LAS}(\pi)$ (the length of the longest $\mathcal{A}$ subsequence in a permutation π) for *long* random permutations π.

Algorithms

What algorithms are there for computing $\mathcal{LAS}(\pi)$?

Algorithms

What algorithms are there for computing $\mathcal{LAS}(\pi)$?

- In one of the early “Theory group” papers we showed that for classes $\mathcal{A}$ defined by a “constructive fixed point equation” (recursive block decomposition) there is a polynomial time algorithm for determining $\mathcal{LAS}(\pi)$.

Algorithms

What algorithms are there for computing $\mathcal{LAS}(\pi)$?

- In one of the early "Theory group" papers we showed that for classes $\mathcal{A}$ defined by a "constructive fixed point equation" (recursive block decomposition) there is a polynomial time algorithm for determining $\mathcal{LAS}(\pi)$.
- Unfortunately, these algorithms are based on dynamic programming on the set $[n] \times [n]$ (and generally on collections of rectangles in this set) so the degrees tend to be rather high. For example, for $\mathrm{Av}(312)$ the complexity is $O(n^5)$.

Algorithms

What algorithms are there for computing $\mathcal{LAS}(\pi)$?

- In one of the early "Theory group" papers we showed that for classes $\mathcal{A}$ defined by a "constructive fixed point equation" (recursive block decomposition) there is a polynomial time algorithm for determining $\mathcal{LAS}(\pi)$.
- Unfortunately, these algorithms are based on dynamic programming on the set $[n] \times [n]$ (and generally on collections of rectangles in this set) so the degrees tend to be rather high. For example, for $\mathrm{Av}(312)$ the complexity is $O(n^5)$.
- Three classes in which we can carry out experiments to some reasonable length:

$$\begin{aligned}\mathcal{L} &= \mathrm{Av}(231, 312) \quad \text{the layered permutations} \\ \mathcal{L}(2) &= \mathrm{Av}(231, 312, 321) \quad \text{layers of size} \leq 2 \\ \mathcal{C} &= \mathrm{Av}(321, 312) \quad \text{direct sums of } 234 \cdots n1\end{aligned}$$

Layered permutations

Complexity of the algorithm is $O(n^2 \log n)$, $s_{\mathcal{L}} = 2$.

Length	μ	σ	$\sim c_{\mathcal{L}}$
1×10^2	23.8	1.8	1.418
2×10^2	34.8	2.2	1.517
4×10^2	50.6	2.5	1.602
8×10^2	73.4	3.0	1.682
16×10^2	105.2	3.3	1.730
32×10^2	150.7	4.0	1.774
64×10^2	215.9	4.4	1.821
128×10^2	307.5	4.9	1.847

Fibonacci ($\mathcal{L}(2)$)

Complexity of the algorithm is $O(n \log n)$ (improvement from theory group paper), $s_{\mathcal{L}(2)} = (1 + \sqrt{5})/2 = 1.618\ldots$.

Length	μ	σ	$\sim c_{\mathcal{L}(2)}$
1×10^4	239.3	4.5	1.431
2×10^4	340.7	5.2	1.451
4×10^4	484.7	6.1	1.468
8×10^4	688.4	6.4	1.481
16×10^4	978.1	7.1	1.495
32×10^4	1386.8	8.3	1.503
64×10^4	1965.3	9.3	1.510
128×10^4	2785.3	10.2	1.515

Sums of cycles

Complexity of the algorithm is $O(n^3 \log n)$ (but in practice better), $s_{\mathcal{C}} = 2$.

Length	μ	σ	$\sim c_{\mathcal{C}}$
1×10^2	22.9	2.0	1.306
2×10^2	33.5	2.3	1.406
4×10^2	48.5	2.4	1.470
8×10^2	70.5	3.1	1.555
16×10^2	101.2	3.3	1.601
32×10^2	145.2	3.9	1.647

Av(312)

Av(312)

- Nothing really known.

Av(312)

- Nothing really known.
- A greedy approach seems to produce a subsequence of length $2\sqrt{n}$ but as that is the same as the longest increasing subsequence that's not very helpful!

Av(312)

- Nothing really known.
- A greedy approach seems to produce a subsequence of length $2\sqrt{n}$ but as that is the same as the longest increasing subsequence that's not very helpful!
- However, the analogue of the greedy approach for the LIS is the *Diversion* that we began with.

Av(312)

- Nothing really known.
- A greedy approach seems to produce a subsequence of length $2\sqrt{n}$ but as that is the same as the longest increasing subsequence that's not very helpful!
- However, the analogue of the greedy approach for the LIS is the *Diversion* that we began with.
- Optimal play in the diversion gives a length of $\sqrt{2n}$, so at least we have some improvement on that.

Av(312)

- Nothing really known.
- A greedy approach seems to produce a subsequence of length $2\sqrt{n}$ but as that is the same as the longest increasing subsequence that's not very helpful!
- However, the analogue of the greedy approach for the LIS is the *Diversion* that we began with.
- Optimal play in the diversion gives a length of $\sqrt{2n}$, so at least we have some improvement on that.
- Optimal play in the diversion extended to avoiding 321 *does not* give $2\sqrt{n}$. Sigh.

Summary

Summary

- Many of the known results for longest increasing subsequences extend to longest $\mathcal{A}$ subsequences.

Summary

- Many of the known results for longest increasing subsequences extend to longest $\mathcal{A}$ subsequences.
- There is an intriguing possibility of a connection between the constants $c_{\mathcal{A}}$ and $s_{\mathcal{A}}$.

Summary

- Many of the known results for longest increasing subsequences extend to longest $\mathcal{A}$ subsequences.
- There is an intriguing possibility of a connection between the constants $c_{\mathcal{A}}$ and $s_{\mathcal{A}}$.
- Classes with good “structural” definitions are the ones in which investigations of $c_{\mathcal{A}}$ are easiest.

Summary

- Many of the known results for longest increasing subsequences extend to longest $\mathcal{A}$ subsequences.
- There is an intriguing possibility of a connection between the constants $c_{\mathcal{A}}$ and $s_{\mathcal{A}}$.
- Classes with good “structural” definitions are the ones in which investigations of $c_{\mathcal{A}}$ are easiest.
- **Wanted**: Good algorithms for finding longest $\mathcal{A}$ subsequences.

Summary

- Many of the known results for longest increasing subsequences extend to longest $\mathcal{A}$ subsequences.
- There is an intriguing possibility of a connection between the constants $c_{\mathcal{A}}$ and $s_{\mathcal{A}}$.
- Classes with good “structural” definitions are the ones in which investigations of $c_{\mathcal{A}}$ are easiest.
- **Wanted**: Good algorithms for finding longest $\mathcal{A}$ subsequences.
- Some interesting aspects of the “online” version of the problem also seem to be emerging.

So long, and thanks for all the fish

Thank you to the organisers of

Permutation Patterns 2006